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SYMMETRY ALGEBRAS OF THE CANONICAL LIE GROUP GEODESIC EQUATIONS IN DIMENSION FIVE

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A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Virginia Commonwealth University.

by

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Nowadays, there is much interest in constructing exact analytical solutions of differential equations using Lie symmetry methods. Lie devised the method in the 1880s. These methods were substantially developed utilizing modern mathematical language in the 1960s and 1970s by several different groups of authors such as L.V. Ovsiannikov, G. Bluman, and P. J. Olver, and have since been implemented as a software package for symbolic computation on commonly used platforms such as Mathematica and MAPLE. In this work, we first develop an algorithmic scheme using the MAPLE platform to perform a Lie symmetry algebra identification and validate it on nonlinear systems of five second-order ordinary differential equations, namely the canonical connection systems of geodesic equations associated with the five-dimensional Lie algebras. In each case, the symmetry generators are determined, and the corresponding nonvanishing Lie brackets are computed. Moreover, the al-

gebraic structure of a Lie algebra of symmetries is identified. Second, we take a theoretical approach and formulate the conditions for a Lie symmetry for the case of the canonical Lie group connection when the Lie algebra is solvable with a one-codimensional abelian nilradical; which pertains to algebras $A_{5,7} - A_{5,18}$. Such conditions have a complicated system of PDEs, as it has several equations and coefficients to be determined. However, we push the integration of such PDEs as much as possible and investigate to what extent they clarify the concrete results obtained by MAPLE. A comparison with qualitative analysis is demonstrated.

CHAPTER 1

INTRODUCTION

I am certain, absolutely certain that... these theories will be recognized as fundamental at some point in the future.—**Sophus Lie**, [1]

Differential Equations, or DEs for short, are an indispensable subject in mathematics, physics, biology, and many other disciplines. They are utilized to model numerous phenomena. For example, fluid draining, meteorology, the spread of infectious diseases, and the behavior of tidal waves all use differential equations. Once a differential equation is given in a question, one is usually concerned with obtaining solutions, whether they are numerical, asymptotic, or analytical. Although many ingenious techniques for obtaining the exact solutions analytically have been developed, the approaches to finding solutions of differential equations can be quite difficult. Specifically, unfamiliar equations whose solutions require unique and creative methods.

1.1 An Overview of Lie Symmetry Analysis of Differential Equations

From the late fifties, Lie group analysis of DEs, also known as Lie symmetry analysis of DEs and Lie symmetry methods of DEs, has been one of the most powerful general techniques for constructing exact analytic solutions of DEs. It was first initiated by the Norwegian mathematician *Marius Sophus Lie*. He was born on December 17, 1842, in Nordfjordied, Norway. Having finished his studies at Nissen's grammar school in Christiania (now Oslo), he matriculated at the University of Christiania to study science in 1861. While studying at the university, he did brilliantly in most subjects, in particular in the mathematical sciences, physics, astronomy, and chemistry. Nevertheless, he did not demonstrate any particular preference for mathematics. The watershed in Lie's relationship to mathematics and his later mathematical career occurred in 1868 after discovering works on modern geometry by Julius Plücker and Jean Victor Poncelet. Following his early research, Lie was awarded a travel scholarship to a half-year in Berlin and a half-year in Paris. While in Berlin, he struck up a friendship and a long fruitful cooperation with German mathematician Felix Klein (1849-1925), who was a student of Plücker. When Lie continued on to Paris in the spring of 1870, Klein came along, and they both enjoyed the regular meetings with French mathematicians, Camile Jordan and Gaston Darboux. The two friends realized the importance of the group concept for the study of geometry. Lie studied the theory of continuous transformation groups and Klein studied discontinuous transformation groups from a geometric standpoint. As a result, they co-published several papers including Klein's *Erlangen Program* [2].

Lie returned to Norway where he instituted the study of continuous transformation groups, which now carry his name as *Lie groups*. He started examining partial differential equations (PDEs), hoping that he could do for differential equations what Galois had done for algebraic equations. Nevertheless, such work was not well-known and accepted in the mathematical community during his time. In September 1884, Klein and Mayer sent their able student, Friedrich Engel, to help Lie in the edition of his work. This collaboration culminated in the publication of three volumes entitled *Theorie der Transformationsgruppen* (The Theory of Transformation Groups) [3, 4, 5] in 1888, 1890, and 1893, respectively. This was Lie's groundbreaking work, with the assistance of Engel. The reference [6] presents a complete bibliography of all Lie's published work.

Sophus Lie passed away on February 18, 1899, and his methods have become tools for understanding and solving problems in numerous areas of science and engineering today. More detailed information on the historical background of Sophus Lie and his work can be found in [6, 7, 8, 9] and the references therein. There are numerous prominent mathematicians not discussed who contributed to his work and those who were, and still are, influenced by his theories.

The application of Lie's symmetry approach to differential equations was not exploited for half a century. Even then, the focus was solely on the abstract theory of Lie groups. This was due to the formidable complexity of computations involved in Lie method. The situation began to change when Ovsiannikov acknowledged the essential application of Lie symmetry method to differential equations in the early sixties. He systematically employed the methods of symmetry analysis of differential equations in the explicit construction of solutions for problems of mathematical physics [11, 10]. Since then, the applications of Lie symmetry analysis has been utilized in numerous problems of differential equations. In 1969, Bluman and Cole [12] reintroduced Lie's method for finding invariant solutions for heat equations. Nevertheless, a formidable obstacle to the generalization of the Bluman and Cole approach were computational problems.

With the advent of reliable symbolic manipulation programs/languages in the eighties, such as LISP, REDUCED, MACSYMA, MATHEMATICA, and MAPLE, the algorithms known to Lie could now be implemented. Since then, the subject has been studied extensively by a number of researchers, which in turn, opened the way to clarification and deeper understanding. Among those are the books by Olver [13], Bluman and Kumei [14], Bluman and Anco [15], Bluman *et al.* [16], Stephani [17], Ibragimov [23, 18, 22, 19, 20, 21], Hydon [24], Hill [25], Cantwell [26], and very recently Arrigo [27].

Since the revival of the application of Lie group theory to DEs, the use of Lie symmetry methods has become an increasingly important part of the study of differential equations, ranging from obtaining new solutions from known ones [13, 14, 28], reducing the order of the given equation [13, 14, 24], deriving conserved quantities [13], to determining whether or not a differential equation can be linearized and construct an explicit linearization when one exists [29, 30, 31]. Moreover, it can be used to classify equations in accordance with their symmetry algebra [33, 32].

To provide a simple explanation, a symmetry is a change or transformation which leaves an object unchanged or invariant. For instance, any rotation of a circle about its center is a symmetry. Various objects can have various degrees of symmetries; intuitively, a circle has more symmetries than a rectangle. Thus, a symmetry can be employed as a classification criterion. In the context of DEs, a symmetry is an invertible transformation of the dependent and independent variables which does not alter the form of the underlying equation or system of equations. Looking at a DE enables one to deduce symmetries, like translations, scalings, and rotations. Certain discrete transformations can also be deduced by inspection. Generally, obtaining all symmetries of DEs is a laborious task that demands an algorithmic approach. If we consider the symmetries of DEs that depend continuously upon a one-parameter group, we can use the algorithm of Lie to compute them. In such an algorithm, the symmetry conditions, also referred to as the determining or defining equations for the symmetries, yield an overdetermined system of linear homogeneous PDEs, upon expansion, which are solved for symmetries.

1.2 The Classification of the Five Dimensional Lie Algebras

For completeness, we present here a succinct summary of the real five-dimensional indecomposable Lie algebras and their classification, as we study the symmetry algebra of their corresponding geodesic equations. Such algebras were classified in 1963 by G.M. Mubarakzyanov [34] and refined in the 1976 paper by J. Patera *et al.* [35].

There are forty classes of five-dimensional Lie algebras listed in [35]. They are denoted as $A^{a}_{p,q}$, which means the qth algebra of dimension p and the superscripts, if any, represent the continuous parameters upon which the algebra depends. The first six algebras are *nilpotent* and do not contain parameters. The algebras $A_{5,7} - A_{5,18}$ are *solvable* and depend on parameters. They have a four-dimensional *abelian nilradical*. The algebras $A_{5,19} - A_{5,32}$ are also *solvable* and depend on parameters, but they have a four-dimensional *non-abelian nilradical*. The remaining algebras $A_{5,33} - A_{5,39}$ are *solvable* and have three-dimensional *nilradicals*. Finally, the algebra $A_{5,40}$ is the only case of dimension five that is *nonsolvable*, and it has a two-dimensional abelian nilradical. See [36, 37, 38] for an excellent exposition of the classification of five-dimensional Lie algebras. The concepts of *Lie algebra*, *nilpotent*, *solvable*, *abelian*, and *nilradical* are defined in Section 2.4.

1.3 Geodesic Equations

The goal of section is to define *Geodesic* and *Geodesic Equations*. The word *Geodesic* originates from the science of geodesy, which is associated with measurements of the Earth's surface [39] (p.163). Moreover, the idea behind the concept of geodesics is the generalization of straight lines in Euclidian space to Riemannian manifolds. For a complete and detailed presentation of geodesics, see the books of McCleary [39], O'Neill [40] and Pressley [41].

Definition 1.1 ([41]). *A curve $\gamma(t)$ on a geometric surface $\mathcal{S} \subset \mathbb{R}^3$ is said to be a **geodesic** if at every point $\gamma(t)$ the acceleration $\ddot{\gamma}(t)$ is either zero or parallel to its unit normal $\mathbf{N}$, where $\mathbf{N}$ is a differentiable Euclidean vector field on $\mathcal{S}$ that has unit length and is everywhere normal to $\mathcal{S}$.*

We next introduce the following example, which strictly follows the definition. The straight line

$$\gamma(t) = ct + d$$

is a geodesic since $\frac{d}{dt}\big(\gamma(t)\big) = c \implies \frac{d^2}{dt^2}\big(\gamma(t)\big) = 0 \iff \ddot{\gamma}(t) = 0.$

Equivalently, the geodesic equations are expressed as a system of nonlinear second-order ODEs [42]

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad (i, j, k = 1, \ldots, n), \tag{1.1}$$

where the coefficients Γ^i_{jk} are the *Christoffel symbols*, which are symmetric in the lower indices $\Gamma^i_{jk} = \Gamma^i_{kj}$, see the book of Hassani [43] for a detailed explanation. Such symbols are defined as

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}), \tag{1.2}$$

where g^{im} is a *metric tensor*, a function that tells how to compute the distance between any two points in a given space. Moreover, they are defined in terms of the components of the connection ∇ as [44]

$$\Gamma^m_{pq} = -\frac{1}{2}\big(Y^j_q X^m_{j,p} + Y^j_q X^m_{j,p}\big), \tag{1.3}$$

Y^j_q are the elements of $n \times n$ matrix and X^m_j is the inverse of Y^j_q. Equations (1.1) represent the trajectories or path of objects moving under gravitational field. They are important in the study of Riemannian Geometry and theoretical physics, especially in General Relativity.

1.4 The Canonical Connection of a Lie Group

The inverse problem for the canonical Lie groups connection leads to so-called the *geodesic equations of the canonical Lie group connection*. The canonical connection, denoted by ∇, was introduced in 1926 by Cartan and Schouten [45]. These geodesic equations are engendered by a Lagrangian function as Euler-Lagrange equations. This issue has been well studied by several different groups of authors. Here, we summarize and synthesize these various accounts.

In the 2003 article of Thompson [46], an investigation was initiated to study the inverse problem for the canonical connection in the case of Lie groups of dimension two and three and, together with Ghanam and Miller, of dimension four [44]. Moreover, Strugar along with Thompson have continued the investigation to dimension five [47]. In [47], they were able to construct a matrix Lie *group* denoted by S corresponding to each five-dimensional Lie algebra listed in the 1976 paper by J. Patera *et al.* [35], as a starting point for the derivation of such geodesics. Then, they constructed the right invariant one-form, by calculating dSS^{-1}, and obtained a basis for the right invariant vector fields, from which the corresponding geodesic equations were determined. For a detailed background, main properties, and an algorithm for such a canonical connection, the reader is referred to [46, 44, 48, 47] and the references given therein.

1.5 Research Objectives

Very recently, Lie symmetry algebra identification has been conducted on a special case of the inverse problem for the geodesic equations pertaining to the canonical connection of Lie groups. Ghanam and Thompson have initiated an investigation of the symmetry algebras of the canonical geodesic equations for Lie groups of dimension two and three [49] as well as four [50] utilizing Lie's methods. They have determined a basis of symmetry algebras for each system of geodesic equations given and have identified their corresponding symmetry algebras. Consequently, the initial goal of this dissertation is to extend these investigations to dimension five. To be more specific, we consider the systems of geodesic equations associated with the five-dimensional indecomposable Lie algebras. Such systems of equations were derived and calculated in [47] by Strugar and Thompson.

Our choice of continuing such an investigation has been motivated by several reasons. First, the situation in dimension five is more complicated than in dimension two, three, and four, as symmetries of the geodesic equations in five-dimensional are of very high dimensionality and involve an intensive computational process. Second, we want to contribute to the understanding of symmetries themselves by studying these large systems of equations since the algorithm of Lie still attract the attention of many researchers and new results are published on a regular basis. Third, we want to discover how to understand large-dimensional systems of vector field Lie algebras and decompose them into some normal categories such as *nilradical*, *solvable complement*, or *semisimple*, as well as identify these Lie algebras as abstract Lie algebras; that is, in some cases, a Lie algebra of symmetries may be isomorphic to a known algebra presented in some accessible list.

The second goal of this dissertation is to take a theoretical approach and formulate the conditions for a Lie symmetry algebra for the case of the canonical Lie group connection when the associated Lie algebra is solvable with a codimension one abelian nilradical. These conditions have a complicated system of PDEs; and it is difficult to be integrated in a complete generality. We, however, push the integration of such PDEs as much as possible and investigate to what extent they explain the concrete results obtained by MAPLE.

1.6 Dissertation Organization

In Chapter 2, we provide the background material, ideas, terminology, and methodology that are used in the subsequent chapters. In particular, we introduce the elements of Lie symmetry analysis of DEs. The published work structuring Chapter 3 is devoted to searching for Lie symmetries and their properties for the geodesic systems of equations associated with the nilpotent Lie algebras. Chapter 4 consists of a published work concerned with Lie symmetries and their properties for the geodesic systems of equations corresponding to the solvable Lie algebras. The materiel belonging to Chapter 5 tackles the analytical derivation of symmetry conditions for the cases whose Lie algebras are of codimension one ablein nilradical. A list of abbreviations and symbols is given in Appendix A.

CHAPTER 2

PRELIMINARIES

Prior to advancing to the main discussion of this work, we present some general results and concepts which are essential for understanding the concept of Lie symmetry methods of differential equations and throughout the whole work. First, we review the concept of groups, one-parameter Lie groups, and later, we discuss the basic concepts of the Lie's theory of symmetry analysis of differential equations and some properties of abstract Lie algebras. Finally, we supply an example illustrating the computation of Lie symmetries of ODEs; we also clarify the symbolic algorithm employed in the identification of Lie symmetry algebras of the geodesic equations under consideration.

2.1 Groups and Lie Groups

In this section, we present a brief review of the Lie groups essential for understanding the concept of Lie symmetry methods of differential equations. For simplicity, we confine our attention to groups, and one-parameter Lie groups of transformations, which will also be helpful in establishing the terminology and the notations commonly used in the subsequent sections. For detailed presentations, see Bluman and Kumei [14], Olver [51], Sattinger and Weaver [52], and Gilmore [53].

2.1.1 Groups

Definition 2.1 ([54])**.** *Let G be a set together with a binary operation, usually called multiplication, that assigns to each ordered pair (g_1, g_2) of elements of G an element in G denoted by g_1g_2. G is called a* ***group*** *under this operation if the following properties* *are satisfied:*

(i) *Closure: For all* $g_1, g_2 \in G$*, then* $g_1 g_2$ *is* $\in G$.

(ii) *Assosiativity: If* $g_1, g_2, g_3 \in G$*, then* $g_1(g_2 g_3) = (g_1 g_2) g_3$.

(iii) *Existence of identity* $I \in G$*:* $g_1 I = I g_1 = g_1$ *for all* $g_1 \in G$.

(iv) *Existence of inverse* $g_1^{-1} \in G$*:* $g_1 g_1^{-1} = g_1^{-1} g_1 = I$ *for all* $g_1 \in G$.

2.1.2 One-Parameter Lie Group of Transformations

One-parameter Lie groups of transformations or just Lie groups are special groups which have an additional property aside from the group properties.

Definition 2.2 ([55, 56])**.** *A set* G *is called a Lie group if*

(i) G *is a group.*

(ii) G *is a smooth manifold. That is, the group operations: the multiplication map*

$$G \times G \to G \text{ defined by } (g_1, g_2) \mapsto g_1 g_2, \tag{2.1}$$

and the inversion map

$$G \to G \text{ defined by } g_1 \mapsto g_1^{-1} \tag{2.2}$$

are smooth functions, meaning that all the derivatives to all orders exist.

It follows from Definition 2.2 that a Lie group carries an algebraic structure of a group, and it is a smooth manifold. The term "smooth manifold" is an object that looks locally like $\mathbb{R}^n$. G is called an n-parameter Lie group if n is the dimension of the manifold. In this dissertation, we focus on one-parameter local Lie groups. The word "local" indicates that Lie groups of transformations are defined in a neighborhood of an identity transformation.

Example 2.1. *The set of complex numbers on the unit circle*

$$S^1 = \{e^{i\theta} : 0 \le \theta < 2\pi\} \subset \mathbb{C} \tag{2.3}$$

is a Lie group with the multiplication operation $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)}$*, and the inverse operation* $e^{(i\theta)^{-1}} = e^{-i\theta}$*. Said differently,* S^1 *is a Lie group since the group multiplication and inversion maps depend smoothly on the parameter* θ*.*

2.2 Symmetries of Differential Equations

This section introduces the mathematical procedures to construct the *symmetries* of a given differential equation. First, we present some definitions of ODEs, which are useful for determining the symmetries, and then we discuss the symmetries of ODEs.

Definition 2.3 ([17])**.** *An nth* $-$ ***order system of ODEs*** *is defined by*

$$F_i\Big(x, y, y', \ldots, y^{(n)}\Big) = 0, \quad (i = 1, \ldots, s), \tag{2.4}$$

where F_i *are functions of* x, y *and the derivatives of* y *up to order* n*,* $y = (y_1, \ldots, y_s)$ *are* s *dependent variables, and* $y', y'', \ldots, y^{(n)}$ *denote the derivatives of* y *with respect to the independent variable* x*.*

Definition 2.4 ([17])**.** *An nth* $-$ ***order ODE*** *defined as*

$$y^{(n)}(x) = F\Big(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\Big), \quad y^{(k)} \equiv \frac{d^k y}{dx^k}, \quad k = 1, \ldots, n, \tag{2.5}$$

where F *is a function of* x, y *and the derivatives of* y *with respect to* x*, can be written as a* ***linear partial differential operator***

$$\Gamma = \Big(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots + F\frac{\partial}{\partial y^{(n-1)}}\Big). \tag{2.6}$$

To illustrate Definition 2.4, consider the ODE $y'' = F(x, y, y') = -y$. Such an ODE can be expressed in terms of the linear operator Γ as $\Gamma = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} - y\frac{\partial}{\partial y'}\right)$.

Dealing with differential equations sometimes requires a suitable change of variables in order for one to simplify the equation. That is, by mapping points (x, y) into points $(\bar{x}, \bar{y})$. Such a technique is called a *point-transformation*.

Definition 2.5 ([17, 55])**.** *A* ***point-transformation*** *is a change of variables*

$$\bar{x} = f(x, y), \quad \bar{y} = g(x, y), \tag{2.7}$$

which is applied to simplify a differential equation under consideration. Here, f and g are functions of the independent variable x and the dependent variable y.

In the study of symmetries of differential equations, one needs to consider transformations on which depend, at least, one arbitrarily continuous parameter ϵ.

Definition 2.6 ([17])**.** *The system (2.4) is said to be invariant under the transformation (Lie group)*

$$\bar{x} = f(x, y, \epsilon), \quad \bar{y} = g(x, y, \epsilon), \tag{2.8}$$

if

$$F_i\big(\bar{x}, \bar{y}, \bar{y}', \ldots, \bar{y}^{(n)}\big) = 0, \quad \textit{whenever} \quad F_i\big(x, y, y', \ldots, y^{(n)}\big) = 0. \tag{2.9}$$

Such a transformation is called a ***symmetry*** *of the differential equation.*

The functions f and g, in Definition 2.6, are smooth functions of the variables x and y. In other words, they are infinitely differentiable with respect to x and y. Furthermore, they are analytic functions in the group parameter ϵ, that is, functions with a convergent Taylor series in ϵ.

Example 2.2. *The Lie group*

$$\bar{x} = e^{\epsilon}x, \quad \bar{y} = e^{\epsilon}y, \tag{2.10}$$

is a symmetry of

$$\frac{dy}{dx} = \frac{y}{x}, \tag{2.11}$$

since

$$\frac{d\bar{y}}{d\bar{x}} = \frac{\frac{d\bar{y}}{dx}}{\frac{d\bar{x}}{dx}} = \frac{e^{\epsilon}\frac{dy}{dx}}{e^{\epsilon}} \implies \frac{d\bar{y}}{d\bar{x}} = \frac{dy}{dx}. \tag{2.12}$$

However, $\frac{dy}{dx} = \frac{y}{x}$, *thus*

$$\frac{d\bar{y}}{d\bar{x}} = \frac{y}{x} = \frac{e^{-\epsilon}\bar{y}}{e^{-\epsilon}\bar{x}} = \frac{\bar{y}}{\bar{x}} \implies \frac{d\bar{y}}{d\bar{x}} = \frac{\bar{y}}{\bar{x}}. \tag{2.13}$$

Now, we define the infinitesimal of groups, which involves expanding the transformation (2.8) in a Taylor series about $\epsilon = 0$.

Definition 2.7 ([57])**.** *A Taylor series expansion of (2.8) about* $\epsilon = 0$*, with* $f(x, y, 0) = x, g(x, y, 0) = y$*, yields the* ***infinitesimal transformation*** *of the group (2.8):*

$$\bar{x} \approx x + \xi(x, y)\epsilon, \quad \bar{y} \approx y + \eta(x, y)\epsilon, \tag{2.14}$$

where

$$\xi(x, y) = \left.\frac{\partial f(x, y, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}, \quad \eta(x, y) = \left.\frac{\partial g(x, y, \epsilon)}{\partial \epsilon}\right|_{\epsilon=0}. \tag{2.15}$$

The partial derivatives of f and g with respect to the group parameter ϵ evaluated at $\epsilon = 0$ are referred to as the ***infinitesimals*** or ***symmetries***. The vector (ξ, η) is defined as the ***vector field*** of the transformation (2.8).

The following definition gives a representation of a one-parameter Lie group of transformation, such as (2.8), in the form of a group generator. This notion will lead us to the discussion of Lie algebras.

Definition 2.8 ([57])**.** *The vector field (2.15) can be written as a first order-differential operator*

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2.16}$$

Such an operator is called the ***infinitesimal generator*** *or* ***symmetry generator*** *of the transformation. It is also called the* ***Lie symmetry vector field*** *of the transformation.*

As an illustration, the group of rotations [57]

$$\overline{x} = x\cos\epsilon + y\sin\epsilon, \quad \overline{y} = -x\sin\epsilon + y\cos\epsilon, \tag{2.17}$$

has the following infinitesimal transformation

$$\bar{x} \approx x + y\epsilon, \quad \bar{y} \approx y - x\epsilon, \tag{2.18}$$

and therefore the group generator (infinitesimal generator) associated with the given Lie group has the form

$$\mathbf{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{2.19}$$

For the sake of clarification, a Lie group G that transforms solutions of a differential equation into solutions are referred to as *symmetries* of the differential equation. Alternatively, the infinitesimal generators of the Lie algebra $\mathfrak{g}$ of such a group G are also referred to as *symmetries* of the differential equation. Moreover, in the literature, this is commonly expressed by saying the differential equation is invariant with respect to G or the differential equation admits G.

2.2.1 Transformation of the Derivatives: Extensions of Infinitesimal Generator

This section aims to clarify how to prolong or extend the infinitesimal generator $\mathbf{X}$ to compute the derivatives $y^{(n)}$ of the variable y in order to apply (2.4). Moreover, such an extension (prolongation) enables us to obtain the symmetries of differential equations of arbitrary order as well as with any number of dependent and independent variables. Define,[17],

$$\left.\begin{aligned}
&\bar{y}' = \tfrac{d\bar{y}}{d\bar{x}} = \tfrac{df(x,y,\epsilon)}{dg(x,y,\epsilon)} = \tfrac{y'(\frac{\partial\bar{y}}{\partial y})+(\frac{\partial\bar{y}}{\partial x})}{y'(\frac{\partial\bar{y}}{\partial y})+(\frac{\partial\bar{y}}{\partial x})} = \bar{y}'(x,y,y',y'',\epsilon),\\
&\bar{y}'' = \tfrac{d\bar{y}'}{d\bar{x}} = \bar{y}''(x,y,y',y'',\epsilon),\\
&\vdots\\
&\bar{y}^{(n)} = \tfrac{d\bar{y}^{(n-1)}}{d\bar{x}} = \bar{y}^{(n-1)}(x,y,y',\ldots,y^{(n-1)},\epsilon).
\end{aligned}\right\} \tag{2.20}$$

The extensions of $\mathbf{X}$ is defined by

$$\left.\begin{aligned}
&\bar{x} = x + \epsilon\xi(x,y) + \ldots,\\
&\bar{y} = y + \epsilon\eta(x,y) + \ldots,\\
&\bar{y}' = y' + \epsilon\eta'(x,y,y') + \ldots,\\
&\vdots\\
&\bar{y}^{(n)} = y^{(n)} + \epsilon\eta^{(n)}(x,y,y',\ldots,y^{(n)}) + \ldots,
\end{aligned}\right\} \tag{2.21}$$

where the expressions $\eta, \eta', \ldots, \eta^{(n)}$ are given by

$$\eta'(x,y,y') = \frac{\partial\bar{y}'}{\partial\epsilon}\bigg|_{\epsilon=0}, \ldots, \eta^{(n)}(x,y,y',\ldots,y^{(n)}) = \frac{\partial\bar{y}^{(n)}}{\partial\epsilon}\bigg|_{\epsilon=0}. \tag{2.22}$$

(2.22) is generalized in a similar manner to (2.15). Substituting the results into (2.21), we obtain

$$\left.\begin{aligned}
&\bar{y}' = y' + \epsilon\eta' + \cdots = \tfrac{d\bar{y}}{d\bar{x}} = \tfrac{dy+\epsilon d\eta+\ldots}{dx+\epsilon d\xi+\ldots} = \tfrac{y'+\epsilon(\frac{d\eta}{dx})+\ldots}{1+\epsilon(\frac{d\xi}{dx})+\ldots} = y' + \epsilon\Big(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\Big),\\
&\vdots\\
&\bar{y}^{(n)} = y^{(n)} + \epsilon\eta^{(n)} + \cdots = y^{(n)} + \epsilon\Big(\frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}\Big) + \ldots,
\end{aligned}\right\} \tag{2.23}$$

from which we obtain

$$\eta^{(n)} = \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}, \tag{2.24}$$

where $\frac{d}{dx}$ is given by

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + +y'' \frac{\partial}{\partial y'} + \cdots + y^{(n)} \frac{\partial}{\partial y^{(n-1)}}. \tag{2.25}$$

Thus,

$$\mathbf{X}^{(n)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta'(x,y,y')\frac{\partial}{\partial y'} + .. + \eta^{(n)}(x,y,y',..,y^{(n)})\frac{\partial}{\partial y^{(n)}}, \tag{2.26}$$

where $\mathbf{X}^{(n)}$ is the n^{th} prolongation of $\mathbf{X}$.

Now, we state the following theorem which characterizes Lie's methods for obtaining symmetries of the differential equations.

Theorem 2.1 ([17])**.** *A system of ODEs (2.4) or an ODE in any of the forms (2.5) or (2.6) admits a group of symmetries with generator (2.26) if and only if*

$$\mathbf{X}^{(n)} F_i = 0, \quad (mod\, F_i = 0), \quad (i = 1, \ldots, s), \tag{2.27}$$

or equivalently,

$$\left[\mathbf{X}^{(n-1)}, \Gamma\right] = \lambda \Gamma, \tag{2.28}$$

holds. λ in (2.28) is not necessarily constant function depending on $\left(x, y, \ldots, y^{(n-1)}\right)$.

Theorem 2.1 tells us that there are two criteria for calculating symmetries. The first criterion is useful in the study of symmetries of low-order ODEs. For $n > 1$, the corresponding identity involves the free variables $y', \ldots, y^{(n-1)}$ allowing to break this identity into a system of linear PDEs, also known as the *defining system* or *determining system* for the symmetries. The second criterion indicates that the symmetries, as generators, form a Lie algebra. In brief, the study of Lie symmetries of a given system of ODEs consists of two essential steps: (i) the determination of symmetry conditions, which the components of the Lie symmetry vector fields must satisfy; and (ii) the solution of the system of these symmetry conditions.

2.3 Lie Algebras

The infinitesimal generators or symmetry generators of the form (2.16) form Lie algebras. They form a closed set with respect to commutation (Lie bracket).

Definition 2.9 ([26]). *The **commutator** or **Lie bracket** of two symmetry generators $\mathbf{X}_a$ and $\mathbf{X}_b$ is the operator generated as follows*

$$\big[\mathbf{X}_a, \mathbf{X}_b\big] = \mathbf{X}_a(\mathbf{X}_b) - \mathbf{X}_b(\mathbf{X}_a). \tag{2.29}$$

For instance, let

$$\mathbf{X}_a = \alpha^j(x)\frac{\partial}{\partial x^j}, \quad \mathbf{X}_b = \beta^k(x)\frac{\partial}{\partial x^k}, \tag{2.30}$$

the commutator or Lie bracket of (2.30) is

$$\begin{aligned} \big[\mathbf{X}_a, \mathbf{X}_b\big] &= \alpha^j(x)\frac{\partial}{\partial x^j}\Big(\beta^k(x)\frac{\partial}{\partial x^k}\Big) - \beta^k(x)\frac{\partial}{\partial x^k}\Big(\alpha^j(x)\frac{\partial}{\partial x^j}\Big) \\ &= \Big(\alpha^j\frac{\partial \beta^k}{\partial x^j}\Big)\frac{\partial}{\partial x^k} - \Big(\beta^k\frac{\partial \alpha^j}{\partial x^k}\Big)\frac{\partial}{\partial x^j}. \end{aligned} \tag{2.31}$$

The operators (2.30) construct a vector space called a Lie algebra. An abstract definition of Lie algebra is presented in the next section.

Definition 2.10 ([26]). *The infinitesimal generators, in the form of (2.16), denoted by $\mathbf{X}_k, k = 1, \ldots, n$, form an **n-dimensional Lie algebra** $\mathfrak{g}$ with following properties:*

(i) The Lie algebra $\mathfrak{g}_n$ is an n-dimensional vector space spanned by the basis set of infinitesimal generators $\mathbf{X}_k, k = 1, \ldots, n$. Let $\alpha, \beta \in \mathbb{R}$ and $\mathbf{X}_a, \mathbf{X}_b, \mathbf{X}_c \in \mathfrak{g}_n$. Thus,

$$\alpha\mathbf{X}_a + \beta\mathbf{X}_b \in \mathfrak{g}_n, \quad \mathbf{X}_a + \mathbf{X}_b = \mathbf{X}_b + \mathbf{X}_a. \tag{2.32}$$

(ii) Bilinearity:

$$\big[\alpha\mathbf{X}_a + \beta\mathbf{X}_b, \mathbf{X}_c\big] = \alpha\big[\mathbf{X}_a, \mathbf{X}_c\big] + \beta\big[\mathbf{X}_b, \mathbf{X}_c\big]. \tag{2.33}$$

Table 1. Commutations of the Lie algebra of (2.37)

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_1$	0	$-\mathbf{X}_3$	$\mathbf{X}_2$
$\mathbf{X}_2$	$\mathbf{X}_3$	0	0
$\mathbf{X}_3$	$-\mathbf{X}_2$	0	0

(iii) Antisymmetry:

$$\big[\mathbf{X}_a, \mathbf{X}_b\big] = -\big[\mathbf{X}_b, \mathbf{X}_a\big] \implies \big(\big[\mathbf{X}_a, \mathbf{X}_a\big] = 0\big). \tag{2.34}$$

(iv) Jacobi identity:

$$\big[\mathbf{X}_a, [\mathbf{X}_b, \mathbf{X}_c]\big] + \big[\mathbf{X}_b, [\mathbf{X}_c, \mathbf{X}_a]\big] + \big[\mathbf{X}_c, [\mathbf{X}_a, \mathbf{X}_b]\big] = 0. \tag{2.35}$$

2.3.1 An Example of Lie Algebra

The following group

$$\bar{x} = x\cos\epsilon_1 - y\sin\epsilon_1 + \epsilon_2, \quad \bar{y} = x\sin\epsilon_1 + y\cos\epsilon_1 + \epsilon_3, \tag{2.36}$$

is called *Group of Rigid Motions* in $\mathbb{R}^2$ [14]. Such a group is a three-parameter Lie group of transformations, where the parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}$. The corresponding infinitesimal generators are

$$\mathbf{X}_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = \frac{\partial}{\partial y}. \tag{2.37}$$

The commutators (Lie brackets) of the infinitesimal generators (2.37) are shown in Table 1. The table is antisymmetric with its diagonal elements all zero, and the structure constants are easily read off from the table, as well as the Jacobi identity is satisfied for (2.37). Thus, the infinitesimal generators (2.37) form a Lie algebra.

2.4 Algebraic Properties of Lie Algebras

After constructing the symmetries of a differential equation under consideration, one can further classify Lie algebras of the symmetries into categories such as *solvable, nilpotent, etc*, that is, to identify their algebraic structures. Therefore, the object of this section is to put together for subsequent chapters some of properties and definitions of abstract Lie algebras. For a fuller introduction to the theory of Lie algebras together with detailed proofs of theorems and propositions, the reader is directed to the books by Jacobson [58], Humphreys [59], Erdmann and Wildon [60], and Knapp [61] as well as references listed herein.

Definition 2.11 (**Lie Algebras**). *Let* $\mathfrak{g}$ *be a vector space over a field* $\mathbb{F}$*. An operation* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, denoted by* $(x, y) \to [x, y]$*, is called the Lie bracket of* x *and* y*. Then* $\mathfrak{g}$ *is called a Lie algebra over* $\mathbb{F}$ *if the following three axioms are satisfied:*

(i) Bilinearity: $\forall a \in \mathbb{F}$ *and* $\forall x, y, z \in \mathfrak{g}$, $[ax + y, z] = a[x, y] + [y, z]$.

(ii) Antisymmetry: $\forall x, y \in \mathfrak{g}$, $[x, y] = -[x, y]$.

(iii) Jacobi identity: $\forall x, y, y \in \mathfrak{g}$, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

The axiom (ii) in Definition 2.11 implies $[x, x] = 0$, $x \in \mathfrak{g}$, if char $\mathbb{F} \neq 2$. $\mathfrak{g}$ is referred to as real if $\mathbb{F} = \mathbb{R}$ and complex if $\mathbb{F} = \mathbb{C}$.

Definition 2.12 (**Lie Subalgebras**). *A subspace* $\mathfrak{h} \subset \mathfrak{g}$ *is a subalgebra of the Lie algebra* $\mathfrak{g}$ *if the Lie bracket or commutator of any two elements of* $\mathfrak{h}$ *is again in* $\mathfrak{h}$*, that is, closed under the commutation* $\left[\mathfrak{h}, \mathfrak{h}\right] \subset \mathfrak{h}$.

Definition 2.13 (**Abelian Lie Algebras**). *A Lie algebra* $\mathfrak{g}$ *is called abelian if the Lie bracket vanishes, that is,* $\left[\mathbf{X}_a, \mathbf{X}_b\right] = 0 \ \forall \mathbf{X}_a, \mathbf{X}_b \in \mathfrak{g}$*, and non-abelian otherwise.*

Definition 2.14 (**Ideal Lie Subalgebras**). *A subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is an ideal if* $\left[\mathfrak{h}, \mathfrak{g}\right] \subset \mathfrak{h}$*, that is, any bracket with elements from* $\mathfrak{g}$ *is in* $\mathfrak{h}$.

Definition 2.15 (**Solvable Lie Algebras**). *Let* $\mathfrak{g}$ *be a Lie algebra. The derived Lie algebra of* $\mathfrak{g}$ *is a subalgebra* $\mathfrak{g}' \equiv \mathfrak{g}^{(1)}$ *defined by* $\mathfrak{g}' = \big[\mathfrak{g}, \mathfrak{g}\big]$*, while the derived series is the sequence of Lie subalgebras defined by*

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}', \ldots, \mathfrak{g}^{(m)} \quad \textit{where} \quad \mathfrak{g}^{(m)} = \big[\mathfrak{g}^{(m-1)}, \mathfrak{g}^{(m-1)}\big],\ m \geq 1.$$

If the derived series eventually arrives at the zero subalgebra, that is, $\mathfrak{g}^{(m)} = 0$ *for some* $m > 0$*, then* $\mathfrak{g}$ *is called a solvable Lie algebra.*

Definition 2.16 (**Nilpotent Lie Algebras**). *Let* $\mathfrak{g}$ *be a Lie algebra. The lower central series of* $\mathfrak{g}$ *is defined by*

$$\mathfrak{g} = \mathfrak{g}^1 \quad \textit{and} \quad \mathfrak{g}^m = [\mathfrak{g}, \mathfrak{g}^{m-1}],\ m \geq 2. \tag{2.38}$$

If the lower central series terminates, that is, $\mathfrak{g}^m = 0$ *for some* $m > 0$*, then* $\mathfrak{g}$ *is called a nilpotent Lie algebra.*

Definition 2.17 (**Radical and Nilradical Lie Algebras**). *The solvable radical or radical of* $\mathfrak{g}$*, denoted by* $\mathfrak{r}$*, is the maximal solvable ideal of* $\mathfrak{g}$*. The nilpotent radical or nilradical of* $\mathfrak{g}$ *is the biggest nilpotent ideal of* $\mathfrak{g}$ *denoted by* $\mathfrak{nr}$*.*

Definition 2.18 (**Decomposable Lie Algebras**). *A Lie algebra* $\mathfrak{g}$ *is called decomposable if it can be decomposed into the direct sum of two or more nonzero Lie algebras, that is,* $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \mathfrak{g}_k$*, and indecomposable otherwise.*

Definition 2.19 (**Simple and Semisimple Lie Algebras**). *A Lie algebra* $\mathfrak{g}$ *is called simple if it is non-abelian and does not have any nontrivial ideal at all. A Lie algebra* $\mathfrak{g}$ *is called semisimple if it is a direct sum of simple Lie algebras.*

Definition 2.20 ([62]). *A solvable Lie algebra* $\mathfrak{g}$ *can be written as the algebraic sum of*

$$\mathfrak{g} = \mathfrak{nr} \dotplus V, \tag{2.39}$$

where $\mathfrak{nr}$ *is the nilradical and* V *is a complementary linear space, that is, a complement to the nilradical* $\mathfrak{nr}$ *in* $\mathfrak{g}$.

The next theorem due to the mathematician Eugenio Elia Levi (1883–1917) [63] asserts that any Lie algebra can be constructed from solvable and semisimple Lie algebras.

Theorem 2.2 (Levi's Theorem). *Any finite-dimensional Lie algebra* $\mathfrak{g}$ *can be decomposed into a vector space direct sum*

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}, \tag{2.40}$$

where $\mathfrak{r}$ *is the radical of* $\mathfrak{g}$ *and* $\mathfrak{s}$ *is a semisimple Lie subalgebra. The semisimple Lie algebra* $\mathfrak{s}$ *is called the Levi factor.*

Definition 2.21 (Semi-direct Product of Lie Algebras). *Let* $\mathfrak{g}$ *be a Lie algebra.* $\mathfrak{g}$ *is the semi-direct product of two subalgebras* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *if the product of* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$ *is a vector space and* $[\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_1$. *We write* $\mathfrak{g} = \mathfrak{g}_2 \rtimes \mathfrak{g}_1$.

Definition 2.22. *In the Levi decomposition the Lie algebra* $\mathfrak{g}$ *is a semi-direct product of* $\mathfrak{s}$ *and* $\mathfrak{r}$ *denoted by* $\mathfrak{s} \rtimes \mathfrak{r}$.

Definition 2.23 (Adjoint Representation). *If* $\mathfrak{g}$ *is a Lie algebra and* $x \in \mathfrak{g}$, *the adjoint representation of* $\mathfrak{g}$ *is defined as* $ad(x)y = [x, y]$, *which provides a matrix representation of* $\mathfrak{g}$.

2.5 An Example of Calculation of Symmetries

In this section, we provide a step-by-step procedure for how to determine the symmetries of the second-order ODE $\dfrac{d^2y}{dx^2} = 0$, construct a Lie algebra of the obtained symmetries and identify its structure, as well as employ the resulting symmetries to solve the ODE. This ODE can be solved by direct integration; however, the basic idea is to illuminate the algorithmic computation of Lie symmetry methods and the usefulness of the recognition of symbolic algorithms presented in this dissertation. It is also crucial to understand these techniques before returning to computer algebra software.

Step one. We let $F = \dfrac{d^2y}{dx^2} = 0$ and the symmetry generator be of the form (2.16), that is,

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{2.41}$$

Step two. We apply the second-order extension of (2.41) to the given ODE since it is of second order. Then, the symmetry condition (2.27) becomes

$$\mathbf{X}^{(2)}F\big|_{F=0} = 0, \tag{2.42}$$

$$\Big(\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y'y''\Big) = 0, \tag{2.43}$$

where $y'' \equiv \dfrac{d^2y}{dx^2}$. Substituting $\dfrac{d^2y}{dx^2} = 0$ into (2.43) gives

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 = 0. \tag{2.44}$$

Step three. We find the system of determining equations by setting the coefficients of y'^3, y'^2, y', and $(y')^0$ to zero. Thus,

$$\xi_{yy} = 0, \tag{2.45}$$

$$\eta_{yy} - 2\xi_{xy} = 0, \tag{2.46}$$

$$2\eta_{xy} - \xi_{xx} = 0, \tag{2.47}$$

$$\eta_{xx} = 0. \tag{2.48}$$

Step four. We solve the system of overdetermined linear PDEs for the functions

$\xi(x, y)$ and $\eta(x, y)$. Integrating (2.45) twice w.r.t. y yields

$$\xi(x, y) = a(x)y + b(x), \tag{2.49}$$

where a and b are arbitrary functions. Integrating (2.48) twice w.r.t. x, we obtain

$$\eta(x, y) = p(y)x + q(y), \tag{2.50}$$

where p and q are further arbitrary functions. Differentiating (2.46) w.r.t. y gives

$$\eta_{yyy} - 2\xi_{xyy} = 0. \tag{2.51}$$

Differentiating (2.45) w.r.t. x and substituting into (2.51) gives

$$\eta_{yyy} = 2\xi_{xyy} \implies \eta_{yyy} = 0, \tag{2.52}$$

from which, we obtain

$$p'''(y)x + q'''(y) = 0. \tag{2.53}$$

When we set the coefficient of x to zero, we get

$$p'''(y) = 0, \tag{2.54}$$

$$q'''(y) = 0. \tag{2.55}$$

The solutions of (2.54) and (2.55) are simply given as

$$p(y) = c_1 + c_2 y + c_{10} y^2, \tag{2.56}$$

$$q(y) = c_4 + c_5 y + c_6 y^2, \tag{2.57}$$

where $c_1, c_2, c_{10}, c_4, c_5,$ and c_6 are constants. Substituting (2.56) and (2.57) into equation (2.50) gives

$$\eta(x,y) = \big(c_1 + c_2 y + c_{10} y^2\big)x + c_4 + c_5 y + c_6 y^2. \tag{2.58}$$

Substituting (2.58) and (2.49) into (2.47), we get

$$2\big(c_2 + 2c_{10} y\big) - \big(a''(x)y + b''(x)\big) = 0. \tag{2.59}$$

Upon setting the coefficients of y to zero, we obtain

$$a''(x) = 4c_{10}, \tag{2.60}$$

$$b''(x) = 2c_2. \tag{2.61}$$

Integrating (2.60) and (2.61) and substituting into (2.49) gives

$$\xi(x,y) = \big(c_7 + c_9 x + 2c_{10} x^2\big)y + c_8 + c_3 x + c_2 x^2. \tag{2.62}$$

Upon substitution of (2.58) and (2.62) into (2.46), we obtain

$$2c_{10}x + 2c_6 - 2\big(c_9 + 4c_{10}x\big) = 0, \tag{2.63}$$

which implies that $c_{10} = 0$ and $c_6 = c_9$. Thus, (2.58) and (2.62) become

$$\xi(x,y) = \big(c_7 + c_6 x\big)y + c_8 + c_3 x + c_2 x^2, \tag{2.64}$$

$$\eta(x,y) = \big(c_1 + c_2 y\big)x + c_4 + c_5 y + c_6 y^2, \tag{2.65}$$

where $c_1 - c_8$ are independent and arbitrary constants.

Step five. We construct a symmetry generator for each constant using the symmetry generator (2.41). The generators will be denoted by $\mathbf{X}_i$, $i = 1, \ldots, 8$, where the coefficients of the generator are obtained by setting $c_i = 1$ and all other constants to

Table 2. The Lie brackets of generators (2.66)

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$
$\mathbf{X}_1$	0	0	$-\mathbf{X}_1$	0	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3 - \mathbf{X}_6$	$-\mathbf{X}_4$
$\mathbf{X}_2$	0	0	$-\mathbf{X}_2$	$-\mathbf{X}_1$	0	0	$-\mathbf{X}_6$	$-2\mathbf{X}_3 - \mathbf{X}_5$
$\mathbf{X}_3$	$\mathbf{X}_1$	$\mathbf{X}_2$	0	0	0	0	$-\mathbf{X}_7$	$-\mathbf{X}_8$
$\mathbf{X}_4$	0	$\mathbf{X}_1$	0	0	$\mathbf{X}_4$	$\mathbf{X}_3 + 2\mathbf{X}_5$	$\mathbf{X}_8$	0
$\mathbf{X}_5$	$-\mathbf{X}_1$	0	0	$-\mathbf{X}_4$	0	$\mathbf{X}_6$	$\mathbf{X}_7$	0
$\mathbf{X}_6$	$-\mathbf{X}_2$	0	0	$-\mathbf{X}_3 - 2\mathbf{X}_5$	$-\mathbf{X}_6$	0	0	$-\mathbf{X}_7$
$\mathbf{X}_7$	$\mathbf{X}_6 - \mathbf{X}_3$	$\mathbf{X}_6$	$\mathbf{X}_7$	$-\mathbf{X}_8$	$-\mathbf{X}_7$	0	0	0
$\mathbf{X}_8$	$\mathbf{X}_4$	$2\mathbf{X}_3 + \mathbf{X}_5$	$\mathbf{X}_8$	0	0	$\mathbf{X}_7$	0	0

zero in ξ and η. This gives

$$\begin{aligned}
&\mathbf{X}_1 = x\frac{\partial}{\partial y}, \qquad \mathbf{X}_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad \mathbf{X}_3 = x\frac{\partial}{\partial x}, \qquad \mathbf{X}_4 = \frac{\partial}{\partial y},\\
&\mathbf{X}_5 = y\frac{\partial}{\partial y}, \qquad \mathbf{X}_6 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \qquad \mathbf{X}_7 = y\frac{\partial}{\partial x}, \qquad \mathbf{X}_8 = \frac{\partial}{\partial x}.
\end{aligned} \tag{2.66}$$

Step six. We find the symmetry algebra of the given ODE by evaluating the brackets of generators of symmetries, and then recognize its Lie algebra type. The Lie brackets of generators (2.66), shown in Table 2, define a semisimple Lie algebra spanned by $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5, \mathbf{X}_6, \mathbf{X}_7, \mathbf{X}_8$. We perform a change of basis a number of times in order to obtain the most standard basis for an abstract semisimple Lie algebra.

$$\begin{aligned}
&e_1 = \mathbf{X}_3 - \mathbf{X}_5, \qquad e_2 = \mathbf{X}_3 + \mathbf{X}_5, \qquad e_3 = 2\mathbf{X}_1, \qquad e_4 = \mathbf{X}_4,\\
&e_5 = \mathbf{X}_7, \qquad e_6 = \mathbf{X}_8, \qquad e_7 = \mathbf{X}_6, \qquad e_8 = 2\mathbf{X}_2.
\end{aligned} \tag{2.67}$$

Evaluation the brackets of (2.67) gives a representation of algebra $\mathfrak{sl}(3, \mathbb{R})$ with its nonvanishing Lie brackets

$$[e_1, e_3] = 2e_3, \quad [e_1, e_4] = e_4, \qquad [e_1, e_5] = -2e_5, \quad [e_1, e_6] = -e_6, \qquad [e_1, e_7] = -e_7,$$

$$[e_1,e_8]=e_8,\quad [e_2,e_4]=-e_4,\quad [e_2,e_6]=-e_6,\quad [e_2,e_7]=e_7,\quad [e_2,e_8]=e_8,$$
$$[e_3,e_5]=2e_1,\quad [e_3,e_6]=-2e_4,\quad [e_3,e_7]=e_8,\quad [e_4,e_5]=e_6,\quad [e_4,e_7]=\frac{e_2}{2}-\frac{e_1}{2},$$
$$[e_4,e_8]=e_3,\quad [e_5,e_8]=2e_7,\quad [e_6,e_7]=e_5,\quad [e_6,e_8]=e_1+3e_2. \tag{2.68}$$

Step seven. We now apply the obtained symmetries (2.66) to find solutions of the given ODE. We will particularly consider $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_6$. We make use of the *Method of Differential Invariants* described in [14]

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)}=\frac{dy'}{\eta'(x,y,y')}=\cdots=\frac{dy^{(n)}}{\eta^{(n)}(x,y,y',\ldots,y^{(n)})}. \tag{2.69}$$

Hence, $\mathbf{X}_1 = x\dfrac{\partial}{\partial y}$. In this case

$$\frac{dy}{x}=0 \implies y=k_1. \tag{2.70}$$

$\mathbf{X}_2 = x^2\dfrac{\partial}{\partial x}+xy\dfrac{\partial}{\partial y}$. In this case

$$\frac{dx}{x^2}=\frac{dy}{xy} \implies \frac{dy}{dx}=\frac{y}{x} \implies y=xe^C \implies y=k_2x. \tag{2.71}$$

$\mathbf{X}_6 = xy\dfrac{\partial}{\partial x}+y^2\dfrac{\partial}{\partial y}$. In this case

$$\frac{dx}{xy}=\frac{dy}{y^2} \implies y=k_3x. \tag{2.72}$$

2.6 Illustration of MAPLE Code-Assisted Identification of Symmetry Algebra

We conclude this chapter by providing readers with a glimpse of how our code is efficiently implemented in the identification of Lie symmetry algebra of a given system of geodesic equations. We present a step-by-step procedure demonstrating the algorithmic scheme written in a MAPLE file. Its essence is to decrease the complexity of the intensive computational process and verify the accuracy of the outcomes. To illuminate such a symbolic algorithm, we consider the geodesics associated with $A^{abc}_{5,7}$.

Implementation of Maple Code

Step A. Begin by clearing the Maple internal memory so that Maple acts as if just booted up.

```
> restart
```

Step B. Initiate all the necessary packages. Use the colon instead of a semicolon to suppress the output.

```
> with(PDEtools) :
> with(DifferentialGeometry) :
> with(LieAlgebras) :
```

Step C. Declare the dependent and independent variables using the commands diff_table and declare to avoid redundancies in the input and in
the display of the output.

> $declare((q, x, y, z, w)(t), (\tau, \lambda, \xi, \eta, \mu, \sigma)(t, q, x, y, z, w))$

q(t) will now be displayed as q

x(t) will now be displayed as x

y(t) will now be displayed as y

z(t) will now be displayed as z

w(t) will now be displayed as w

tau(t, q, x, y, z, w) will now be displayed as τ

lambda(t, q, x, y, z, w) will now be displayed as λ

xi(t, q, x, y, z, w) will now be displayed as ξ

eta(t, q, x, y, z, w) will now be displayed as η

mu(t, q, x, y, z, w) will now be displayed as μ

sigma(t, q, x, y, z, w) will now be displayed as σ **(1)**

> $DepVars := [q, x, y, z, w](t);$

$$DepVars := [q, x, y, z, w] \quad \textbf{(2)}$$

> $Q, X, Y, Z, W := diff\_table(q(t)), diff\_table(x(t)), diff\_table(y(t)), diff\_table(z(t)), diff\_table(w(t)) :$

Step D. Declare the system of ODEs (geodesics) corresponding to $A_{5,7}$ i.e.,

$q^{\cdot\cdot}(t) = q^{\cdot}(t)w^{\cdot}(t),\ x^{\cdot\cdot}(t) = ax^{\cdot}(t)w^{\cdot}(t),\ y^{\cdot\cdot}(t) = b\,y^{\cdot}(t)w^{\cdot}(t),\ z^{\cdot\cdot}(t) = cz^{\cdot}(t)w^{\cdot}(t),\ w^{\cdot\cdot}(t) = 0.$

> $abc \neq 0, -1 \leq c \leq b \leq a \leq 1$

$$abc \neq 0, -1 \leq c \textbf{ and } c \leq b \textbf{ and } b \leq a \leq 1 \quad \textbf{(3)}$$

> $Eq1 := Q_{t,t} = Q_t \cdot W_t$

$$Eq1 := q_{t,t} = (q_t)\,(w_t) \quad \textbf{(4)}$$

> $Eq2 := X_{t,t} = a \cdot X_t \cdot W_t$

$$Eq2 := x_{t,t} = a\left(x_t\right)\left(w_t\right) \tag{5}$$

> $Eq3 := Y_{t,t} = b \cdot Y_t \cdot W_t$

$$Eq3 := y_{t,t} = b\left(y_t\right)\left(w_t\right) \tag{6}$$

> $Eq4 := Z_{t,t} = c \cdot Z_t \cdot W_t$

$$Eq4 := z_{t,t} = c\left(z_t\right)\left(w_t\right) \tag{7}$$

> $Eq5 := W_{t,t} = 0$

$$Eq5 := w_{t,t} = 0 \tag{8}$$

> $PDESYS := [Eq1, Eq2, Eq3, Eq4, Eq5]$

$$PDESYS := \left[q_{t,t} = \left(q_t\right)\left(w_t\right), x_{t,t} = a\left(x_t\right)\left(w_t\right), y_{t,t} = b\left(y_t\right)\left(w_t\right), z_{t,t} = c\left(z_t\right)\left(w_t\right), w_{t,t} = 0\right] \tag{9}$$

Step E. Compute the infinitesimals of symmetry generators and determine the number of symmetries for the given system. Use the Infinitesimals and nops commands directly.

> $G := Infinitesimals(PDESYS);\ nops(G);$

$$\begin{aligned}G := \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 1\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 1, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = t, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 1, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 1, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 1, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 1, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = q, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = z\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = w, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\\ \Big[&\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = x, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0\Big],\end{aligned}$$

$\left[\_\xi_t(t,q,w,x,y,z)=0, \_\eta_q(t,q,w,x,y,z)=0, \_\eta_w(t,q,w,x,y,z)=0, \_\eta_x(t,q,w,x,y,z)=0, \_\eta_y(t,q,w,x,y,z)=y, \_\eta_z(t,q,w,x,y,z)=0\right], \left[\_\xi_t(t,q,w,x,y,z)=0, \_\eta_q(t,q,w,x,y,z)=e^w, \_\eta_w(t,q,w,x,y,z)=0, \_\eta_x(t,q,w,x,y,z)=0, \_\eta_y(t,q,w,x,y,z)=0, \_\eta_z(t,q,w,x,y,z)=0\right], \left[\_\xi_t(t,q,w,x,y,z)=0, \_\eta_q(t,q,w,x,y,z)=0, \_\eta_w(t,q,w,x,y,z)=0, \_\eta_x(t,q,w,x,y,z)=0, \_\eta_y(t,q,w,x,y,z)=0, \_\eta_z(t,q,w,x,y,z)=e^{cw}\right], \left[\_\xi_t(t,q,w,x,y,z)=0, \_\eta_q(t,q,w,x,y,z)=0, \_\eta_w(t,q,w,x,y,z)=0, \_\eta_x(t,q,w,x,y,z)=e^{aw}, \_\eta_y(t,q,w,x,y,z)=0, \_\eta_z(t,q,w,x,y,z)=0\right], \left[\_\xi_t(t,q,w,x,y,z)=0, \_\eta_q(t,q,w,x,y,z)=0, \_\eta_w(t,q,w,x,y,z)=0, \_\eta_x(t,q,w,x,y,z)=0, \_\eta_y(t,q,w,x,y,z)=e^{bw}, \_\eta_z(t,q,w,x,y,z)=0\right]$

```
Error, invalid input: nops expects 1 argument, but received 16
```

Step F. Rewrite the symmetry vector fields in a simple and efficient way.

> **for** *i* **from** 1 **to** 16 **do**

$\text{Gamma}[i] := evalDG(rhs(G[i][1])\cdot D\_t + rhs(G[i][2])\cdot D\_q + rhs(G[i][3])\cdot D\_w + rhs(G[i][4])\cdot D\_x + rhs(G[i][5])\cdot D\_y + rhs(G[i][6])\cdot D\_z)$

od

$$\Gamma_1 := D\_z$$

$$\Gamma_2 := D\_w$$

$$\Gamma_3 := t\,D\_t$$

$$\Gamma_4 := D\_t$$

$$\Gamma_5 := D\_q$$

$$\Gamma_6 := D\_x$$

$$\Gamma_7 := D\_y$$

$$\Gamma_8 := q\,D\_q$$

$$\Gamma_9 := z\,D\_z$$

$$\Gamma_{10} := w\,D\_t$$

$$\Gamma_{11} := x\,D\_x$$

$$\Gamma_{12} := y\,D\_y$$

$$\Gamma_{13} := e^w\,D\_q$$

$$\Gamma_{14} := e^{cw}\,D\_z$$

$$\Gamma_{15} := e^{a\,w}\, D\_x$$

$$\Gamma_{16} := e^{b\,w}\, D\_y \qquad \textbf{(10)}$$

> Gamma := $evalDG$([seq(Gamma[i], i = 1 ..16)]); $nops$(Gamma)

$$\Gamma := [D\_z, D\_w, t\,D\_t, D\_t, D\_q, D\_x, D\_y, q\,D\_q, z\,D\_z, w\,D\_t, x\,D\_x, y\,D\_y, e^{w}\,D\_q, e^{c\,w}\,D\_z, e^{a\,w}\,D\_x, e^{b\,w}\,D\_y]$$

$$16 \qquad \textbf{(11)}$$

Step G. Build the Lie algebra structure for the Lie algebra of symmetry vector fields Gamma. The output is a list of nonzero Lie brackets.

> $DGsetup$([t, x, y, z, w, q], M);

frame name: M **(12)**

M > g := $LieAlgebraData$(Gamma, $Alg1$);

$$g := [[e1, e9] = e1, [e2, e10] = e4, [e2, e13] = e13, [e2, e14] = c\,e14, [e2, e15] = a\,e15, [e2, e16] = b\,e16, [e3, e4] = -e4, [e3, e10] = -e10, [e5, e8] = e5, [e6, e11] = e6, [e7, e12] = e7, [e8, e13] = -e13, [e9, e14] = -e14, [e11, e15] = -e15, [e12, e16] = -e16] \qquad \textbf{(13)}$$

Step H. Store these nonzero Lie brackets in memory using the command DGsetup.

M > $DGsetup$(g)

Lie algebra: Alg1 **(14)**

Step I. Check whether or not the Lie algebra g is indecomposable.

Alg1 > $Query$("Indecomposable");

true **(15)**

Step J. Identify the type of g, i.e., solvable, nilpotent, semisimple, or none.

Alg1 > $Query$("Solvable");

true **(16)**

Alg1 > $Query$("Nilpotent");

false **(17)**

Alg1 > $Query$("Semisimple");

false **(18)**

Step K. Determine the nilradical of g, i.e. the biggest nilpotent ideal of g, since it is solvable, and verify whether or not it is abelian.

Alg1 > Nil := $Nilradical$( ); $nops$(Nil);

$$Nil := [e1, e4, e5, e6, e7, e10, e13, e14, e15, e16]$$

(19)

$$10 \quad \textbf{(19)}$$

Alg1 > $Query(Nil, \text{"Subalgebra"});$

$$true \quad \textbf{(20)}$$

Alg1 > $Query(Nil, \text{"Abelian"});$

$$true \quad \textbf{(21)}$$

Alg1 > $LieAlgebraData(Nil);$

$$[\;] \quad \textbf{(22)}$$

Step L. Find a complement of Nil and verify whether or not it is abelian.

Alg1 > $COMP := [e2, e3, e8, e9, e11, e12];$

$$COMP := [e2, e3, e8, e9, e11, e12] \quad \textbf{(23)}$$

Alg1 > $Query(COMP, \text{"Subalgebra"});$

$$true \quad \textbf{(24)}$$

Alg1 > $Query(COMP, \text{"Abelian"});$

$$true \quad \textbf{(25)}$$

Alg1 > $LieAlgebraData(COMP)$

$$[\;] \quad \textbf{(26)}$$

Step M. Repeat Steps A to F.

> $restart$

> $with(PDEtools):\ with(DifferentialGeometry):\ with(LieAlgebras):$

> $declare\big((q, x, y, z, w)(t), (\tau, \lambda, \xi, \eta, \mu, \sigma)(t, q, x, y, z, w)\big)$

$q(t)$ *will now be displayed as* q

$x(t)$ *will now be displayed as* x

$y(t)$ *will now be displayed as* y

$z(t)$ *will now be displayed as* z

$w(t)$ *will now be displayed as* w

$tau(t, q, x, y, z, w)$ *will now be displayed as* τ

$lambda(t, q, x, y, z, w)$ *will now be displayed as* λ

$xi(t, q, x, y, z, w)$ *will now be displayed as* ξ

$eta(t, q, x, y, z, w)$ *will now be displayed as* η

$mu(t, q, x, y, z, w)$ *will now be displayed as* μ

$sigma(t, q, x, y, z, w)$ *will now be displayed as* σ **(27)**

> $DepVars := [q, x, y, z, w](t);$

$$DepVars := [q, x, y, z, w] \quad \textbf{(28)}$$

> $Q, X, Y, Z, W := diff\_table(q(t)), diff\_table(x(t)), diff\_table(y(t)), diff\_table(z(t)), diff\_table(w(t)):$

> $abc \neq 0, -1 \leq c \leq b \leq a \leq 1$

$$abc \neq 0,\ -1 \leq c \text{ and } c \leq b \text{ and } b \leq a \leq 1 \tag{29}$$

> $Eq1 := Q_{t,t} = Q_t \cdot W_t$

$$Eq1 := q_{t,t} = (q_t)(w_t) \tag{30}$$

> $Eq2 := X_{t,t} = a \cdot X_t \cdot W_t$

$$Eq2 := x_{t,t} = a(x_t)(w_t) \tag{31}$$

> $Eq3 := Y_{t,t} = b \cdot Y_t \cdot W_t$

$$Eq3 := y_{t,t} = b(y_t)(w_t) \tag{32}$$

> $Eq4 := Z_{t,t} = c \cdot Z_t \cdot W_t$

$$Eq4 := z_{t,t} = c(z_t)(w_t) \tag{33}$$

> $Eq5 := W_{t,t} = 0$

$$Eq5 := w_{t,t} = 0 \tag{34}$$

> $PDESYS := [Eq1, Eq2, Eq3, Eq4, Eq5]$

$$PDESYS := [q_{t,t} = (q_t)(w_t), x_{t,t} = a(x_t)(w_t), y_{t,t} = b(y_t)(w_t), z_{t,t} = c(z_t)(w_t), w_{t,t} = 0] \tag{35}$$

> $G := Infinitesimals(PDESYS);\ nops(G);$

$G := [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 1], [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 1, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = t, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = 1, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 1, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 1, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 1, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = q, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = z], [\_\xi_t(t,q,w,x,y,z) = w, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,x,y,z) = 0, \_\eta_x(t,q,w,x,y,z) = 0, \_\eta_y(t,q,w,x,y,z) = 0, \_\eta_z(t,q,w,x,y,z) = 0], [\_\xi_t(t,q,w,x,y,z) = 0, \_\eta_q(t,q,w,x,y,z) = 0, \_\eta_w(t,q,w,$

$x, y, z) = 0, \_\eta_x(t, q, w, x, y, z) = x, \_\eta_y(t, q, w, x, y, z) = 0, \_\eta_z(t, q, w, x, y, z) = 0\big],$
$\big[\_\xi_t(t, q, w, x, y, z) = 0, \_\eta_q(t, q, w, x, y, z) = 0, \_\eta_w(t, q, w, x, y, z) = 0, \_\eta_x(t, q, w, x, y, z)$
$= 0, \_\eta_y(t, q, w, x, y, z) = y, \_\eta_z(t, q, w, x, y, z) = 0\big], \big[\_\xi_t(t, q, w, x, y, z) = 0, \_\eta_q(t, q, w,$
$x, y, z) = e^{w}, \_\eta_w(t, q, w, x, y, z) = 0, \_\eta_x(t, q, w, x, y, z) = 0, \_\eta_y(t, q, w, x, y, z) = 0, \_\eta_z(t,$
$q, w, x, y, z) = 0\big], \big[\_\xi_t(t, q, w, x, y, z) = 0, \_\eta_q(t, q, w, x, y, z) = 0, \_\eta_w(t, q, w, x, y, z) = 0,$
$\_\eta_x(t, q, w, x, y, z) = 0, \_\eta_y(t, q, w, x, y, z) = 0, \_\eta_z(t, q, w, x, y, z) = e^{c\,w}\big], \big[\_\xi_t(t, q, w, x,$
$y, z) = 0, \_\eta_q(t, q, w, x, y, z) = 0, \_\eta_w(t, q, w, x, y, z) = 0, \_\eta_x(t, q, w, x, y, z) = e^{a\,w}, \_\eta_y(t,$
$q, w, x, y, z) = 0, \_\eta_z(t, q, w, x, y, z) = 0\big], \big[\_\xi_t(t, q, w, x, y, z) = 0, \_\eta_q(t, q, w, x, y, z) = 0,$
$\_\eta_w(t, q, w, x, y, z) = 0, \_\eta_x(t, q, w, x, y, z) = 0, \_\eta_y(t, q, w, x, y, z) = e^{b\,w}, \_\eta_z(t, q, w, x, y,$
$z) = 0\big]$

Error, invalid input: nops expects 1 argument, but received 16

> for i **from** 1 **to** 16 **do**

$\text{Gamma}[i] := evalDG(rhs(G[i][1]) \cdot D\_t + rhs(G[i][2]) \cdot D\_q + rhs(G[i][3]) \cdot D\_w + rhs(G[i][4]) \cdot D\_x + rhs(G[i][5]) \cdot D\_y + rhs(G[i][6]) \cdot D\_z)$

od

$$\Gamma_1 := D\_z$$
$$\Gamma_2 := D\_w$$
$$\Gamma_3 := t\,D\_t$$
$$\Gamma_4 := D\_t$$
$$\Gamma_5 := D\_q$$
$$\Gamma_6 := D\_x$$
$$\Gamma_7 := D\_y$$
$$\Gamma_8 := q\,D\_q$$
$$\Gamma_9 := z\,D\_z$$
$$\Gamma_{10} := w\,D\_t$$
$$\Gamma_{11} := x\,D\_x$$
$$\Gamma_{12} := y\,D\_y$$
$$\Gamma_{13} := e^{w}\,D\_q$$
$$\Gamma_{14} := e^{c\,w}\,D\_z$$

$$\Gamma_{15} := e^{a\,w}\, D\_x$$

$$\Gamma_{16} := e^{b\,w}\, D\_y \tag{36}$$

> Gamma := $evalDG$([seq(Gamma[i], i = 1 ..16)]); $nops$(Gamma);

$\Gamma := [D\_z, D\_w, t\, D\_t, D\_t, D\_q, D\_x, D\_y, q\, D\_q, z\, D\_z, w\, D\_t, x\, D\_x, y\, D\_y, e^{w}\, D\_q, e^{c\,w}\, D\_z, e^{a\,w}\, D\_x, e^{b\,w}\, D\_y]$

$$16 \tag{37}$$

Step N. Perform a change of basis to the symmetry vector fields Gamma so that the Nil and COMP can be written on a basis resembling the basis (*e1,...,e10*) and (*e11,..,e16*), respectively. Keep performing a change of basis until you arrive at (*e1,...,e10*) a basis for Nil and (*e11,...,e16*) for a complement of Nil. Then repeat Steps G to K.

> Xi := [Gamma[1], Gamma[16], Gamma[15], Gamma[4], Gamma[5], Gamma[6], Gamma[7], Gamma[14], Gamma[13], Gamma[10], Gamma[11], Gamma[12], Gamma[9], Gamma[8], Gamma[3], Gamma[2]]; $nops$(Xi);

$\Xi := [D\_z, e^{b\,w}\, D\_y, e^{a\,w}\, D\_x, D\_t, D\_q, D\_x, D\_y, e^{c\,w}\, D\_z, e^{w}\, D\_q, w\, D\_t, x\, D\_x, y\, D\_y, z\, D\_z, q\, D\_q, t\, D\_t, D\_w]$

$$16 \tag{38}$$

> $DGsetup$([t, x, y, z, w, q], M);

frame name: M **(39)**

M > g := $LieAlgebraData$(Xi, $Alg1$);

$g := [[e1, e13] = e1, [e2, e12] = e2, [e2, e16] = -b\, e2, [e3, e11] = e3, [e3, e16] = -a\, e3, [e4, e15] = e4, [e5, e14] = e5, [e6, e11] = e6, [e7, e12] = e7, [e8, e13] = e8, [e8, e16] = -c\, e8, [e9, e14] = e9, [e9, e16] = -e9, [e10, e15] = e10, [e10, e16] = -e4]$ **(40)**

M > $DGsetup$(g)

Lie algebra: Alg1 **(41)**

Alg1 > $Query$("Indecomposable");

true **(42)**

Alg1 > $Query$("Solvable");

true **(43)**

Alg1 > $Query$("Nilpotent");

false **(44)**

Alg1 > $Query$("Semisimple");

false **(45)**

Alg1 > Nil := $Nilradical$( ); $nops$(Nil);

$Nil := [e1, e2, e3, e4, e5, e6, e7, e8, e9, e10]$

$$10 \tag{46}$$

Alg1 > $Query$(Nil, "Subalgebra");

true **(47)**

```
Alg1 > Query(Nil, "Abelian");
```
true **(48)**

```
Alg1 > LieAlgebraData(Nil);
```
[ ] **(49)**

```
Alg1 > COMP := [e11, e12, e13, e14, e15, e16]; nops(COMP);
```
$COMP := [e11, e12, e13, e14, e15, e16]$

6 **(50)**

```
Alg1 > Query(COMP, "Subalgebra");
```
true **(51)**

```
Alg1 > Query(COMP, "Abelian");
```
true **(52)**

```
Alg1 > LieAlgebraData(COMP);
```
[ ] **(53)**

Conclusion. The symmetry algebra is a sixteen-dimensional indecomposable solvable. It has a ten-dimensional abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and a six-dimensional abelian complement spanned by $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$. The symmetry algebra as a whole is isomorphic to $\mathbb{R}^6 \ltimes \mathbb{R}^{10}$.

CHAPTER 3

SYMMETRIES OF THE CANONICAL GEODESIC EQUATIONS OF FIVE-DIMENSIONAL NILPOTENT CASES

In this chapter, we investigate the Lie symmetry properties of the geodesic systems of five-dimensional indecomposable nilpotent Lie groups whose associated Lie algebras are listed in [35]. Such algebras are mutually not isomorphic and furthermore, unlike many of the low-dimensional solvable Lie algebras, are "sporadic", in the sense that they do not belong to continuous families that depend on parameters. The corresponding geodesic systems of equations for each of the nilpotent Lie groups were constructed in [47]. We devote a separate section to each case. For each case, we methodically provide the nonzero brackets of the original Lie algebra, the associated system of geodesics, a basis for the associated Lie algebra of symmetries, and the corresponding nonvanishing Lie brackets. We then proceed to identify each Lie symmetry algebra, noting whether it is solvable, semisimple, or neither, and, in the latter case, give the semi-direct sum of semisimple and solvable algebras.

It turns out that of the six nilpotent Lie algebras that are considered, two have flat canonical connections; that is, where the right-hand side of the corresponding geodesics is zero. Indeed, in both cases we provide a change of coordinates so that the geodesic equations describe the motion of a "free particle". However, it is to be emphasized that such a change of coordinates is *not* compatible with the Lie algebra structure; that is, the Lie algebra of the symmetries of a free particle system is isomorphic to $\mathfrak{sl}(n+2,\mathbb{R})$. Nonetheless, it follows that in these two cases the Lie symmetry algebra must be $\mathfrak{sl}(7,\mathbb{R})$.

One final qualitative remark is in order. We observe that, roughly speaking, the dimensions of the Lie symmetry algebras of the corresponding systems of geodesic equations of nilpotent Lie algebras are larger than that of comparable solvable algebras, which are investigated in Chapter 4. Indeed, we observe that two cases, considered in Sections 3.2 and 3.5, lead to flat connections and so provide symmetry algebras of maximal dimension. We believe that there may be two underlying reasons that help to explain this phenomenon. First of all, the nilpotent algebras, at least in dimension five, do not depend on parameters. Secondly, it appears as though the geodesic systems for nilpotent Lie algebras, always contain several trivial geodesic equations, that is, where the right-hand side is zero. We believe that this circumstance deserves to be further investigated.

An additional point to emphasize is that determining the symmetry algebra basis and identifying its Lie algebraic structure in each of these cases, in this chapter and the subsequent chapter, constitutes a major challenge. The intensive computational process is facilitated and verified by the MAPLE symbolic manipulation program illuminated in Section 2.6. Throughout this chapter and the following chapters, (q, x, y, z, w) are the dependent variables, and $(\dot{q}, \dot{x}, \dot{y}, \dot{z}, \dot{w})$ denote the first order derivatives of (q, x, y, z, w) with respect to t. Additionally, the variables (q, x, y, z, w) and their dots represent the position coordinates and the corresponding velocities coordinates. Finally, we use, for example, shorthand D_r for $\frac{\partial}{\partial r}$ to denote a coordinate vector field.

3.1 Lie Symmetries of Free Particle Systems

We first review the Lie symmetries of a free particle system, being the most extreme case of a flat connection. The same results have been rediscovered many times, but we refer to [64] as one source. The geodesic equations will be written as

$$\ddot{x}^i = 0, \quad i = 1, \ldots, n, \tag{3.1}$$

where x^i are a system of local coordinates on some manifold M. It is helpful to define the dilation vector field Δ on M by

$$\Delta = tD_t + x^i D_i, \tag{3.2}$$

where D_t are D_i denote the partial derivative operator with respect to t and x^i, respectively, and there is a sum over i from 1 to n, the latter being the dimension of M. Then, the following vector fields comprise a standard basis for the space of Lie symmetries of (3.1) [65]:

$$D_t,\ D_i,\ tD_t,\ x^i D_t,\ tD_i,\ x^i D_j,\ t\Delta,\ x^i\Delta. \tag{3.3}$$

Adding up, we obtain a space of dimension $n^2 + 4n + 3 = (n+2)^2 - 1$ and indeed we obtain a representation of the simple Lie algebra $\mathfrak{sl}(n+2, \mathbb{R})$. Put differently, amongst all $(n^2 + 4n + 3)$-dimensional Lie algebras solely $\mathfrak{sl}(n + 2, \mathbb{R})$ can be isomorphic to the symmetry algebra of (3.1), where n is the dimension of the system. This result is standard and long known. Lie proved such a result for $n = 1$ [66].

3.2 Algebra$_{5,1}$

In this section, we study the Lie symmetry algebra of geodesic equations associated with the Lie algebra $A_{5,1}$, whose nonzero brackets are listed [35]. As stated in the introduction to this chapter and the previous chapter, the associated system of geodesic equations of this case, $A_{5,1}$, and all remaining cases, $A_{5,2} - A_{5,6}$, was derived in [47]. Accordingly, for each case, we methodically present the nonzero brackets of the original Lie algebra and the associated system of geodesics. Then, we find a basis of symmetry generators and evaluate their Lie brackets. Sequentially, we proceed to identify each Lie symmetry algebra.

The nonzero brackets of the original Lie algebra $A_{5,1}$ are

$$[e_3, e_5] = e_1,\ [e_4, e_5] = e_2, \tag{3.4}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{y}\dot{w}, \quad \ddot{x} = \dot{z}\dot{w}, \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.5}$$

We make a change of variable to equations $\ddot{q}$ and $\ddot{x}$ so the system can be written as the free particle system. Thus,

$$\bar{q} = q - \frac{1}{2}yw \implies \ddot{\bar{q}} = 0, \quad \text{and} \quad \bar{x} = x - \frac{1}{2}zw \implies \ddot{\bar{x}} = 0. \tag{3.6}$$

Hence, (3.5) become

$$\ddot{\bar{q}} = 0, \quad \ddot{\bar{x}} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.7}$$

The symmetry Lie algebra is $\mathfrak{sl}(7, \mathbb{R})$ as clarified in Section 3.1.

3.3 Algebra$_{5,2}$

In this section, we consider the associated geodesics of algebra $A_{5,2}$. Its nonzero brackets are

$$[e_2, e_5] = e_1,\ [e_3, e_5] = e_2,\ [e_4, e_5] = e_3, \tag{3.8}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{x}\dot{w}, \quad \ddot{x} = \dot{y}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.9}$$

We implement our symbolic algorithms described in Section 2.6 to the system (3.9). Hence, we obtain the following basis of symmetry generators

$$\left.\begin{aligned}
&e_1 = D_t,\ e_2 = tD_q,\ e_3 = D_q,\ e_4 = D_y,\ e_5 = D_x,\ e_6 = D_z,\\
&e_7 = D_w + \tfrac{xD_q+yD_x+zDy}{2},\ e_8 = zD_t, e_9 = zD_q,\ e_{10} = wD_q,\\
&e_{11} = wD_t,\ e_{12} = yD_q + zD_x,\ e_{13} = twD_q + 2tD_x,\\
&e_{14} = \tfrac{1}{2}w^2D_q + wD_x,\ e_{15} = wzD_q + 2zD_x,\\
&e_{16} = (zw-2y)D_t,\ e_{17} = \tfrac{1}{6}w^3D_q + \tfrac{1}{2}w^2D_x + wD_y,\\
&e_{18} = \tfrac{1}{24}w^4D_q + \tfrac{1}{6}w^3D_x + \tfrac{1}{2}w^2D_y + wD_z,\\
&e_{19} = (yw - \tfrac{1}{2}zw^2)D_q + (2y-zw)D_x,\ e_{20} = tD_t,\\
&e_{21} = qD_q + xD_x + yD_y + zD_z,\\
&e_{22} = \tfrac{w}{24}(w^3D_q + 4w^2D_x + 12wD_y + 24D_z),\\
&e_{23} = \tfrac{(2y-wz)}{2}(2D_x + wD_q),\\
&e_{24} = \tfrac{(w^3z-3w^2y+6wx-12q)}{6}D_q + \tfrac{(wz-2y)}{2}(2D_y + wD_x),\\
&e_{25} = (\tfrac{zw^4}{24} + \tfrac{xw^2}{2} - \tfrac{yw^3}{6} - qw)D_q + (\tfrac{zw^3}{6} + wx - \tfrac{yw^2}{2} - 2q)D_x\\
&+(\tfrac{zw^2}{2} - yw)D_y + (zw-2y)D_z.
\end{aligned}\right\} \tag{3.10}$$

Evaluation of the brackets of (3.10) gives a solvable Lie algebra $\mathfrak{g}_A$ with the following nonvanishing Lie brackets

$[e_1, e_2] = e_3,$	$[e_1, e_{13}] = e_{10} + 2e_5,$	$[e_1, e_{20}] = e_1,$
$[e_2, e_8] = -e_9,$	$[e_2, e_8] = -e_9,$	$[e_2, e_{11}] = -e_{10},$
$[e_2, e_{16}] = 2e_{12} - e_{15},$	$[e_2, e_{20}] = -e_2,$	$[e_2, e_{21}] = e_2,$
$[e_2, e_{22}] = 2e_2,$	$[e_2, e_{24}] = -e_2,$	$[e_2, e_{25}] = -e_{13},$
$[e_3, e_{21}] = e_3,$	$[e_3, e_{22}] = 2e_3,$	$[e_3, e_{24}] = -e_3,$

$$[e_3, e_{25}] = -e_{10} - 2e_5, \qquad [e_4, e_7] = \frac{e_5}{2}, \qquad [e_4, e_{12}] = e_3,$$

$$[e_4, e_{16}] = -2e_1, \qquad [e_4, e_{19}] = e_{10} + 2e_5, \qquad [e_4, e_{21}] = e_4,$$

$$[e_4, e_{22}] = -\frac{e_{14}}{2}, \qquad [e_4, e_{23}] = -\frac{e_{10}}{2}, \qquad [e_4, e_{24}] = -e_{14} - e_4,$$

$$[e_4, e_{25}] = -e_{17} - 2e_6, \qquad [e_5, e_7] = \frac{e_3}{2}, \qquad [e_5, e_{21}] = e_5,$$

$$[e_5, e_{22}] = \frac{e_{10}}{2} + 2e_5, \qquad [e_5, e_{23}] = e_3, \qquad [e_5, e_{24}] = e_{10} + e_5,$$

$$[e_5, e_{25}] = e_{14}, \qquad [e_6, e_7] = \frac{e_4}{2}, \qquad [e_6, e_8] = e_1,$$

$$[e_6, e_9] = e_3, \qquad [e_6, e_{12}] = e_5, \qquad [e_6, e_{15}] = e_{10} + 2e_5,$$

$$[e_6, e_{16}] = e_{11}, \qquad [e_6, e_{19}] = -e_{14}, \qquad [e_6, e_{21}] = e_6,$$

$$[e_6, e_{22}] = \frac{e_{17}}{2}, \qquad [e_6, e_{23}] = e_4 + \frac{e_{14}}{2}, \qquad [e_6, e_{24}] = e_{17} + e_6,$$

$$[e_6, e_{25}] = e_{18}, \qquad [e_7, e_{10}] = e_3, \qquad [e_7, e_{11}] = e_1,$$

$$[e_7, e_{14}] = e_5 + \frac{e_{10}}{2}, \qquad [e_7, e_{17}] = e_4 + \frac{e_{14}}{2}, \qquad [e_7, e_{18}] = e_6 + \frac{e_{17}}{2},$$

$$[e_7, e_{22}] = e_7, \qquad [e_8, e_{13}] = e_{15}, \qquad [e_8, e_{18}] = -e_{11},$$

$$[e_8, e_{20}] = e_8, \qquad [e_8, e_{21}] = -e_8, \qquad [e_8, e_{24}] = -e_8,$$

$$[e_8, e_{25}] = -e_{16}, \qquad [e_9, e_{18}] = -e_{10}, \qquad [e_9, e_{22}] = 2e_9,$$

$$[e_9, e_{24}] = -2e_9, \qquad [e_9, e_{25}] = 2e_{12} - 2e_{15}, \qquad [e_{10}, e_{21}] = e_{10},$$

$$[e_{10}, e_{22}] = e_{10}, \qquad [e_{10}, e_{24}] = -e_{10}, \qquad [e_{10}, e_{25}] = -2e_{14},$$

$$[e_{11}, e_{13}] = 2e_{14}, \qquad [e_{11}, e_{20}] = e_{11}, \qquad [e_{11}, e_{22}] = -e_{11},$$

$$[e_{12}, e_{17}] = -e_{10}, \qquad [e_{12}, e_{18}] = -e_{14}, \qquad [e_{12}, e_{22}] = 2e_{12},$$

$$[e_{13}, e_{16}] = 2e_{19}, \qquad [e_{13}, e_{20}] = -e_{13}, \qquad [e_{13}, e_{21}] = e_{13},$$

$$[e_{13}, e_{22}] = 2e_{13}, \qquad [e_{13}, e_{23}] = 2e_2, \qquad [e_{13}, e_{24}] = e_{13},$$

$$[e_{14}, e_{21}] = e_{14}, \qquad [e_{14}, e_{22}] = e_{14}, \qquad [e_{14}, e_{23}] = e_{10},$$

$$[e_{14}, e_{24}] = e_{14}, \qquad [e_{15}, e_{18}] = -2e_{14}, \qquad [e_{15}, e_{22}] = 2e_{15},$$

$$
\begin{array}{lll}
[e_{15}, e_{23}] = 2e_9, & [e_{15}, e_{25}] = 2e_{19}, & [e_{16}, e_{17}] = 2e_{11}, \\
[e_{16}, e_{20}] = e_{16}, & [e_{16}, e_{21}] = -e_{16}, & [e_{16}, e_{23}] = 2e_8, \\
[e_{16}, e_{24}] = e_{16}, & [e_{17}, e_{19}] = 2e_{14}, & [e_{17}, e_{21}] = e_{17}, \\
[e_{17}, e_{22}] = -e_{17}, & [e_{17}, e_{24}] = -e_{17}, & [e_{17}, e_{25}] = -2e_{18}, \\
[e_{18}, e_{21}] = e_{18}, & [e_{18}, e_{22}] = -e_{18}, & [e_{18}, e_{23}] = e_{17}, \\
[e_{18}, e_{24}] = e_{18}, & [e_{19}, e_{22}] = 2e_{19}, & [e_{19}, e_{23}] = 2e_{12} - 2e_{15}, \\
[e_{19}, e_{24}] = 2e_{19}, & &
\end{array}
$$

and a semisimple Lie algebra with the following nonvanishing Lie brackets

$$
[e_{23}, e_{24}] = -2e_{23}, \qquad [e_{23}, e_{25}] = -2e_{24}, \qquad [e_{24}, e_{25}] = -2e_{25}. \tag{3.11}
$$

$\mathfrak{g}_A$ is spanned by the symmetry generators e_1 through e_{22}. It is solvable since the derived series eventually arrives at zero subalgebra, that is,

$$
\mathfrak{g}_A = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{20}, e_{21}, e_{22}\}, \tag{3.12}
$$

$$
\begin{aligned}
\mathfrak{g}_A^{(1)} = \{e_3, e_{10} + 2e_5, e_1, -e_9, -e_{10}, 2e_{12} - e_{15}, -e_2, e_4, -\frac{e_{14}}{2}, e_{11}, e_6, \frac{e_{17}}{2}, \\
e_7, e_{15}, e_8, 2e_{19}, -e_{13}, e_{16}, e_{18}\},
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
\mathfrak{g}_A^{(2)} = \{-e_3, -e_{10} - 2e_5, e_9, e_{10}, -2e_{12} + e_{15}, -2e_1, -2e_{14}, \frac{e_4}{2}, e_{11}, \\
\frac{e_{17}}{2} + e_6, -e_{15}, -2e_{19}\},
\end{aligned} \tag{3.14}
$$

$$
\mathfrak{g}_A^{(3)} = \{-e_3, -e_{10} - 2e_5\}, \tag{3.15}
$$

$$
\mathfrak{g}_A^{(4)} = \{0\}. \tag{3.16}
$$

The nilradical of $\mathfrak{g}_A$, denoted by $\mathbb{R}^{19}$, is non-abelian and spanned by e_1, e_2, e_3, e_4, $e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}$. The complement to the nilradical $\mathbb{R}^3$ is abelian and spanned by e_{20}, e_{21}, e_{22}. $\mathfrak{g}_A$ is a semi-direct product of $\mathbb{R}^3$ and $\mathbb{R}^{19}$ written as $\mathbb{R}^3 \rtimes \mathbb{R}^{19}$. (3.11) gives the standard Lie brackets of the algebra $\mathfrak{sl}(2,\mathbb{R})$ spanned by e_{23}, e_{24}, e_{25}. We conclude that the symmetry algebra of the system (3.9) is a twenty-five-dimensional indecomposable nontrivial Levi decomposition algebra $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathbb{R}^{19})$.

3.4 Algebra$_{5,3}$

In this section, we consider the associated geodesics of algebra $A_{5,3}$. Its nonzero brackets are

$$[e_3, e_4] = e_2,\ [e_3, e_5] = e_1,\ [e_4, e_5] = e_3, \tag{3.17}$$

and the associated system of geodesic equations is

$$\ddot{q} = -2z\dot{z}\dot{w}, \quad \ddot{x} = 2\dot{y}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.18}$$

By the procedures described in Section 2.6, the symmetry generators of (3.18) are

$$\begin{aligned}
e_1 = D_t,\ e_2 = tD_q,\ e_3 = D_q,\ e_4 = tD_x,\ e_5 = D_x, e_6 = D_y,\ e_7 = D_w,\\
e_8 = zD_t,\ e_9 = wD_q,\ e_{10} = zD_q,\ e_{11} = zD_x,\ e_{12} = wD_x,\ e_{13} = wD_t,\\
e_{14} = yD_x + \frac{zD_y}{2},\ e_{15} = zwD_x + zD_y,\ e_{16} = -zwD_q + D_z,\\
e_{17} = \frac{w^2D_x}{2} + \frac{wD_y}{2},\ e_{18} = (y - \frac{zw}{2})D_q,\\
e_{19} = tD_t + (yw - \frac{w^2z}{2})D_x + (y - \frac{zw}{2})D_y,\\
e_{20} = qD_q + 2xD_x + yD_y + wD_w,\\
e_{21} = tD_t - (yw - \frac{w^2z}{2})D_x - (y - \frac{zw}{2})D_y, e_{22} = twD_x + tD_y,\\
e_{23} = (zw - 2y)D_t,
\end{aligned}$$

$$e_{24} = qD_q - wD_w - xD_x + zD_z,$$
$$e_{25} = zD_w - \frac{z^3 D_q}{3} + \frac{z^2 D_y}{2} + (yz - \frac{q}{2})D_x,$$
$$e_{26} = (2yw - 2x - w^2 z)D_q + \frac{w^3 D_x}{3} + \frac{w^2 D_y}{2} + wD_z. \qquad (3.19)$$

Their nonvanishing Lie brackets define the following structure of Lie algebra $\mathfrak{g}$

$[e_1, e_2] = e_3,$ $[e_1, e_4] = e_5,$ $[e_1, e_{19}] = e_1,$

$[e_1, e_{21}] = e_1,$ $[e_1, e_{22}] = e_{12} + e_6,$ $[e_2, e_8] = -e_{10},$

$[e_2, e_{13}] = -e_9,$ $[e_2, e_{19}] = -e_2,$ $[e_2, e_{20}] = 3e_2,$

$[e_2, e_{21}] = -e_2,$ $[e_2, e_{23}] = 2e_{18},$ $[e_2, e_{24}] = e_2,$

$[e_2, e_{25}] = -\frac{e_4}{2},$ $[e_3, e_{20}] = 3e_3,$ $[e_3, e_{24}] = e_3,$

$[e_3, e_{25}] = -\frac{e_5}{2},$ $[e_4, e_8] = -e_{11},$ $[e_4, e_{13}] = -e_{12},$

$[e_4, e_{19}] = -e_4,$ $[e_4, e_{20}] = 3e_4,$ $[e_4, e_{21}] = -e_4,$

$[e_4, e_{23}] = 2e_{14} - e_{15},$ $[e_4, e_{24}] = -e_4,$ $[e_4, e_{26}] = -2e_2,$

$[e_5, e_{20}] = 3e_5,$ $[e_5, e_{24}] = -e_5,$ $[e_5, e_{26}] = -2e_3,$

$[e_6, e_{14}] = e_5,$ $[e_6, e_{18}] = e_3,$ $[e_6, e_{19}] = e_{12} + e_6,$

$[e_6, e_{20}] = -2e_{12},$ $[e_6, e_{21}] = -e_{12} - e_6,$ $[e_6, e_{23}] = -2e_1,$

$[e_6, e_{25}] = e_{11},$ $[e_6, e_{26}] = 2e_9,$ $[e_7, e_9] = e_3,$

$[e_7, e_{12}] = e_5,$ $[e_7, e_{13}] = e_1,$ $[e_7, e_{15}] = e_{11},$

$[e_7, e_{16}] = -e_{10},$ $[e_7, e_{17}] = e_{12} + \frac{e_6}{2},$ $[e_7, e_{18}] = -\frac{e_{10}}{2},$

$[e_7, e_{19}] = e_{14} - e_{15},$ $[e_7, e_{20}] = 2e_{15} - 2e_{14} + e_7,$ $[e_7, e_{21}] = -e_{14} + e_{15},$

$[e_7, e_{22}] = e_4,$ $[e_7, e_{23}] = e_8,$ $[e_7, e_{24}] = -e_7,$

$[e_7, e_{26}] = 2e_{18} + e_{16} + 2e_{17},$ $[e_8, e_{16}] = -e_1,$ $[e_8, e_{19}] = e_8,$

$[e_8, e_{20}] = -e_8,$ $[e_8, e_{21}] = e_8,$ $[e_8, e_{22}] = e_{15},$

$$[e_8, e_{24}] = -e_8, \qquad [e_8, e_{26}] = -e_{13}, \qquad [e_9, e_{20}] = 2e_9,$$

$$[e_9, e_{24}] = -2e_9, \qquad [e_9, e_{25}] = -e_{10} - \frac{e_{12}}{2}, \qquad [e_{10}, e_{16}] = -e_3,$$

$$[e_{10}, e_{20}] = 2e_{10}, \qquad [e_{10}, e_{25}] = -\frac{e_{11}}{2}, \qquad [e_{10}, e_{26}] = -e_9,$$

$$[e_{11}, e_{16}] = -e_5, \qquad [e_{11}, e_{20}] = 2e_{11}, \qquad [e_{11}, e_{24}] = -2e_{11},$$

$$[e_{11}, e_{26}] = -2e_{10} - e_{12}, \qquad [e_{12}, e_{20}] = 2e_{12}, \qquad [e_{12}, e_{25}] = -e_{11},$$

$$[e_{12}, e_{26}] = -2e_9, \qquad [e_{13}, e_{19}] = e_{13}, \qquad [e_{13}, e_{20}] = -e_{13},$$

$$[e_{13}, e_{21}] = e_{13}, \qquad [e_{13}, e_{22}] = 2e_{17}, \qquad [e_{13}, e_{24}] = e_{13},$$

$$[e_{13}, e_{25}] = -e_8, \qquad [e_{14}, e_{15}] = -e_{11}, \qquad [e_{14}, e_{16}] = -\frac{e_6}{2},$$

$$[e_{14}, e_{17}] = -\frac{e_{12}}{2}, \qquad [e_{14}, e_{18}] = \frac{e_{10}}{2}, \qquad [e_{14}, e_{19}] = -e_{14} + e_{15},$$

$$[e_{14}, e_{20}] = 3e_{14} - 2e_{15}, \qquad [e_{14}, e_{21}] = e_{14} - e_{15}, \qquad [e_{14}, e_{22}] = -e_4,$$

$$[e_{14}, e_{23}] = -e_8, \qquad [e_{14}, e_{24}] = -e_{14}, \qquad [e_{14}, e_{26}] = -e_{17} - 2e_{18},$$

$$[e_{15}, e_{16}] = -e_{12} - e_6, \qquad [e_{15}, e_{18}] = e_{10}, \qquad [e_{15}, e_{19}] = e_{15},$$

$$[e_{15}, e_{20}] = -e_{15}, \qquad [e_{15}, e_{21}] = -e_{15}, \qquad [e_{15}, e_{23}] = -2e_8,$$

$$[e_{15}, e_{24}] = -e_{15}, \qquad [e_{15}, e_{26}] = -2e_{17}, \qquad [e_{16}, e_{18}] = -\frac{e_9}{2},$$

$$[e_{16}, e_{19}] = -e_{17}, \qquad [e_{16}, e_{20}] = e_{16} + 2e_{17}, \qquad [e_{16}, e_{21}] = e_{17},$$

$$[e_{16}, e_{23}] = e_{13}, \qquad [e_{16}, e_{24}] = e_{16}, \qquad [e_{16}, e_{25}] = \frac{e_{15}}{2} + e_{14} + e_7,$$

$$[e_{17}, e_{18}] = \frac{e_9}{2}, \qquad [e_{17}, e_{19}] = e_{17}, \qquad [e_{17}, e_{20}] = -e_{17},$$

$$[e_{17}, e_{21}] = -e_{17}, \qquad [e_{17}, e_{23}] = -e_{13}, \qquad [e_{17}, e_{24}] = e_{17},$$

$$[e_{17}, e_{25}] = -\frac{e_{15}}{2}, \qquad [e_{18}, e_{19}] = -e_{18}, \qquad [e_{18}, e_{20}] = 3e_{18},$$

$$[e_{18}, e_{21}] = e_{18}, \qquad [e_{18}, e_{22}] = -e_2, \qquad [e_{18}, e_{24}] = e_{18},$$

$$[e_{18}, e_{25}] = -\frac{e_{14}}{2} + \frac{e_{15}}{2}, \qquad [e_{21}, e_{22}] = 2e_{22}, \qquad [e_{21}, e_{23}] = -2e_{23},$$

$$[e_{22}, e_{23}] = -2e_{21}, \qquad [e_{24}, e_{24}] = 2e_{25}, \qquad [e_{24}, e_{26}] = -2e_{26},$$

$[e_{25}, e_{26}] = e_{24}$.

The solvable part of $\mathfrak{g}$ is spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18} e_{19}, e_{20}$, the nilradical is non-abelian and spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}$. The complement to the nilradical is abelian and spanned by e_{19}, e_{20}. The semisimple part has two copies of $\mathfrak{sl}(2, \mathbb{R})$ spanned by e_{21}, e_{22}, e_{23} and e_{24}, e_{25}, e_{26}. Thus, the symmetry algebra is a twenty-six-dimensional indecomposable nontrivial Levi decomposition algebra.

3.5 Algebra$_{5,4}$

In this section, we consider the associated geodesics of algebra $A_{5,4}$. Its nonzero brackets are

$$[e_2, e_4] = e_1, \ [e_3, e_5] = e_1, \tag{3.20}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{y}\dot{w} + \dot{x}\dot{z}, \quad \ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.21}$$

Setting $\bar{q} = q - \frac{1}{2}(yw + xz) \implies \ddot{\bar{q}} = 0$, then (3.21) becomes

$$\ddot{\bar{q}} = 0, \quad \ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.22}$$

The symmetry Lie algebra is $\mathfrak{sl}(7, \mathbb{R})$.

3.6 Algebra$_{5,5}$

In this section, we consider the associated geodesics of algebra $A_{5,5}$. The nonzero brackets are

$$[e_3, e_4] = e_1, \ [e_2, e_5] = e_1, \ [e_3, e_5] = e_2, \tag{3.23}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{x}\dot{w} + \dot{y}\dot{z}, \quad \ddot{x} = \dot{y}\dot{w}, \quad \ddot{y} = 0, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.24}$$

Applying the algorithms of Section 2.6, we find the symmetry algebra basis

$$\left.\begin{aligned}
&e_1 = D_y,\ e_2 = D_t,\ e_3 = D_x,\ e_4 = tD_q,\ e_5 = D_q,\ e_6 = D_z,\ e_7 = D_w,\\
&e_8 = wD_t,\ e_9 = yD_t,\ e_{10} = zD_q,\ e_{11} = yD_q,\ e_{12} = wD_q,\\
&e_{13} = \tfrac{w^2D_q}{2} + wD_x,\ e_{14} = \tfrac{y^2D_q}{2} + yD_z,\ e_{15} = \tfrac{wyD_q}{2} + yD_x,\\
&e_{16} = \tfrac{wyD_q}{2} + wD_z,\ e_{17} = (x - \tfrac{wy}{2})D_q,\\
&e_{18} = \tfrac{w(w^2+3z)D_q}{6} + \tfrac{w^2D_x}{2} + wD_y,\\
&e_{19} = tD_t + \tfrac{yzD_q}{2} + zD_z - \tfrac{w(wy-2x)D_q}{4} + (x - \tfrac{wy}{2})D_x,\\
&e_{20} = (\tfrac{yw^2}{4} - \tfrac{xw}{2} + q)D_q + \tfrac{wyD_x}{2} + yD_y,\\
&e_{21} = (\tfrac{w^2y}{4} - zy - \tfrac{xw}{2} + 2q)D_q + wD_w + \tfrac{wyD_x}{2},\\
&e_{22} = tD_t + \tfrac{(w^2y-2xw)D_q}{4} - (x - \tfrac{wy}{2})D_x,\\
&e_{23} = zD_t, e_{24} = \tfrac{tyD_q}{2} + tD_z,\ e_{25} = \tfrac{twD_q}{2} + tD_x,\\
&e_{26} = (wy - 2x)D_t,\ e_{27} = \tfrac{wzD_q}{2} + zD_x,\\
&e_{28} = \tfrac{yzD_q}{2} + zD_z + \tfrac{(w^2y-2xw)D_q}{4} - (x - \tfrac{wy}{2})D_x,\\
&e_{29} = (\tfrac{y^2w}{2} - xy)D_q + (wy - 2x)D_z.
\end{aligned}\right\} \tag{3.25}$$

and their nonvanishing Lie brackets are,

$$\begin{array}{lll}
[e_1, e_9] = e_2, & [e_1, e_{11}] = e_5, & [e_1, e_{14}] = e_{11} + e_6,\\
[e_1, e_{15}] = e_3 + \dfrac{e_{12}}{2}, & [e_1, e_{16}] = \dfrac{e_{12}}{2}, & [e_1, e_{17}] = -\dfrac{e_{12}}{2},\\
[e_1, e_{19}] = \dfrac{e_{10}}{2} - \dfrac{e_{13}}{2}, & [e_1, e_{20}] = e_1 + \dfrac{e_{13}}{2}, & [e_1, e_{21}] = -e_{10} + \dfrac{e_{13}}{2},\\
[e_1, e_{22}] = \dfrac{e_{13}}{2}, & [e_1, e_{24}] = \dfrac{e_4}{2}, & [e_1, e_{26}] = e_8,
\end{array}$$

$$[e_1, e_{28}] = \frac{e_{10}}{2} + \frac{e_{13}}{2}, \qquad [e_1, e_{29}] = e_{16} - e_{17}, \qquad [e_2, e_4] = e_5,$$

$$[e_2, e_{19}] = e_2, \qquad [e_2, e_{22}] = e_2, \qquad [e_2, e_{24}] = e_6 + \frac{e_{11}}{2},$$

$$[e_2, e_{25}] = e_3 + \frac{e_{12}}{2}, \qquad [e_3, e_{17}] = e_5, \qquad [e_3, e_{19}] = e_3 + \frac{e_{12}}{2},$$

$$[e_3, e_{20}] = -\frac{e_{12}}{2}, \qquad [e_3, e_{21}] = -\frac{e_{12}}{2}, \qquad [e_3, e_{22}] = -\frac{e_{12}}{2} - e_3,$$

$$[e_3, e_{26}] = -2e_2, \qquad [e_3, e_{28}] = -\frac{e_{12}}{2} - e_3, \qquad [e_3, e_{29}] = -e_{11} - 2e_6,$$

$$[e_4, e_8] = -e_{12}, \qquad [e_4, e_9] = -e_{11}, \qquad [e_4, e_{19}] = -e_4,$$

$$[e_4, e_{20}] = e_4, \qquad [e_4, e_{21}] = 2e_4, \qquad [e_4, e_{22}] = -e_4,$$

$$[e_4, e_{23}] = -e_{10}, \qquad [e_4, e_{26}] = 2e_{17}, \qquad [e_5, e_{20}] = e_5,$$

$$[e_5, e_{21}] = 2e_5, \qquad [e_6, e_{10}] = e_5, \qquad [e_6, e_{18}] = \frac{e_{12}}{2},$$

$$[e_6, e_{19}] = e_6 + \frac{e_{11}}{2}, \qquad [e_6, e_{21}] = -e_{11}, \qquad [e_6, e_{23}] = e_2,$$

$$[e_6, e_{27}] = e_3 + \frac{e_{12}}{2}, \qquad [e_6, e_{28}] = e_6 + \frac{e_{11}}{2}, \qquad [e_7, e_8] = e_2,$$

$$[e_7, e_{12}] = e_5, \qquad [e_7, e_{13}] = e_{12} + e_3, \qquad [e_7, e_{15}] = \frac{e_{11}}{2},$$

$$[e_7, e_{16}] = e_6 + \frac{e_{11}}{2}, \qquad [e_7, e_{17}] = -\frac{e_{11}}{2}, \qquad [e_7, e_{18}] = e_{13} + e_1 + \frac{e_{10}}{2},$$

$$[e_7, e_{19}] = -\frac{e_{15}}{2} + \frac{e_{17}}{2}, \qquad [e_7, e_{20}] = \frac{e_{15}}{2} - \frac{e_{17}}{2}, \qquad [e_7, e_{21}] = -\frac{e_{17}}{2} + e_7 + \frac{e_{15}}{2},$$

$$[e_7, e_{22}] = \frac{e_{15}}{2} - \frac{e_{17}}{2}, \qquad [e_7, e_{25}] = \frac{e_4}{2}, \qquad [e_7, e_{26}] = e_9,$$

$$[e_7, e_{27}] = \frac{e_{10}}{2}, \qquad [e_7, e_{28}] = \frac{e_{15}}{2} - \frac{e_{17}}{2}, \qquad [e_7, e_{29}] = e_{14},$$

$$[e_8, e_{19}] = e_8, \qquad [e_8, e_{21}] = -e_8, \qquad [e_8, e_{22}] = e_8,$$

$$[e_8, e_{24}] = e_{16}, \qquad [e_8, e_{25}] = e_{13}, \qquad [e_9, e_{18}] = -e_8,$$

$$[e_9, e_{19}] = e_9, \qquad [e_9, e_{20}] = -e_9, \qquad [e_9, e_{22}] = e_9,$$

$$[e_9, e_{24}] = e_{14}, \qquad [e_9, e_{25}] = e_{15}, \qquad [e_{10}, e_{14}] = -e_{11},$$

$$[e_{10}, e_{16}] = -e_{12}, \qquad [e_{10}, e_{19}] = -e_{10}, \qquad [e_{10}, e_{20}] = e_{10},$$

$$[e_{10}, e_{21}] = 2e_{10}, \qquad [e_{10}, e_{24}] = -e_4, \qquad [e_{10}, e_{28}] = -e_{10},$$

$[e_{10}, e_{29}] = 2e_{17},$ $[e_{11}, e_{18}] = -e_{12},$ $[e_{11}, e_{21}] = 2e_{11},$

$[e_{12}, e_{20}] = e_{12},$ $[e_{12}, e_{21}] = e_{12},$ $[e_{13}, e_{17}] = e_{12},$

$[e_{13}, e_{19}] = e_{13},$ $[e_{13}, e_{21}] = -e_{13},$ $[e_{13}, e_{22}] = -e_{13},$

$[e_{13}, e_{26}] = -2e_{8},$ $[e_{13}, e_{28}] = -e_{13},$ $[e_{13}, e_{29}] = -2e_{16},$

$[e_{14}, e_{18}] = -e_{16},$ $[e_{14}, e_{19}] = e_{14},$ $[e_{14}, e_{20}] = -e_{14},$

$[e_{14}, e_{23}] = e_{9},$ $[e_{14}, e_{27}] = e_{15},$ $[e_{14}, e_{28}] = e_{14},$

$[e_{15}, e_{17}] = e_{11},$ $[e_{15}, e_{18}] = -e_{13},$ $[e_{15}, e_{19}] = e_{15},$

$[e_{15}, e_{20}] = -e_{15},$ $[e_{15}, e_{22}] = -e_{15},$ $[e_{15}, e_{26}] = -2e_{9},$

$[e_{15}, e_{28}] = -e_{15},$ $[e_{15}, e_{29}] = -2e_{14},$ $[e_{16}, e_{19}] = e_{16},$

$[e_{16}, e_{21}] = -e_{16},$ $[e_{16}, e_{23}] = e_{8},$ $[e_{16}, e_{27}] = e_{13},$

$[e_{16}, e_{28}] = e_{16},$ $[e_{17}, e_{19}] = -e_{17},$ $[e_{17}, e_{20}] = e_{17},$

$[e_{17}, e_{21}] = 2e_{17},$ $[e_{17}, e_{22}] = e_{17},$ $[e_{17}, e_{25}] = -e_{4},$

$[e_{17}, e_{27}] = -e_{10},$ $[e_{17}, e_{28}] = e_{17},$ $[e_{18}, e_{20}] = e_{18},$

$[e_{18}, e_{21}] = -e_{18},$ $[e_{22}, e_{23}] = -e_{23},$ $[e_{22}, e_{24}] = e_{24},$

$[e_{22}, e_{25}] = 2e_{25},$ $[e_{22}, e_{26}] = -2e_{26},$ $[e_{22}, e_{27}] = e_{27},$

$[e_{22}, e_{29}] = -e_{29},$ $[e_{23}, e_{24}] = -e_{22} + e_{28},$ $[e_{23}, e_{25}] = e_{27},$

$[e_{23}, e_{28}] = -e_{23},$ $[e_{23}, e_{29}] = -e_{26},$ $[e_{24}, e_{26}] = -e_{29},$

$[e_{24}, e_{27}] = e_{25},$ $[e_{24}, e_{28}] = e_{24},$ $[e_{25}, e_{26}] = -2e_{22},$

$[e_{25}, e_{28}] = -e_{25},$ $[e_{25}, e_{29}] = -2e_{24},$ $[e_{26}, e_{27}] = 2e_{23},$

$[e_{26}, e_{28}] = e_{26},$ $[e_{27}, e_{28}] = -2e_{27},$ $[e_{27}, e_{29}] = -2e_{28},$

$[e_{28}, e_{29}] = -2e_{29}.$

The symmetry algebra is a twenty-nine-dimensional with a nontrivial Levi decomposition. The semisimple part is $\mathfrak{sl}(3,\mathbb{R})$ with basis $e_{22}, e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{28}, e_{29}$. The radical is a twenty-one-dimensional with an eighteen-dimensional nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}$ and an abelian complement spanned by e_{19}, e_{20}, e_{21}.

3.7 Algebra$_{5,6}$

In this section, we consider the associated geodesics of algebra $A_{5,6}$. The nonzero brackets are

$$[e_3, e_4] = e_1,\ [e_2, e_5] = e_1,\ [e_3, e_5] = e_2,\ [e_4, e_5] = e_3, \tag{3.26}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = 2\dot{x}\dot{w} - z\dot{z}\dot{w}, \quad \ddot{x} = \dot{y}\dot{w}, \quad \ddot{y} = \dot{z}\dot{w}, \quad \ddot{z} = 0, \quad \ddot{w} = 0. \tag{3.27}$$

By algorithmic means described in Section 2.6, the symmetry algebra basis is

$$\begin{aligned}
e_1 = D_t,\ e_2 = D_x,\ e_3 = D_y,\ e_4 = tD_q,\ e_5 = D_q,\ e_6 = D_w,\ e_7 = zD_q,\\
e_8 = wD_q,\ e_9 = wD_t,\ e_{10} = zD_t,\ e_{11} = twD_q + tD_x,\ e_{12} = w^2D_q + wD_x,\\
e_{13} = 2xD_q + yD_x + zD_y,\ e_{14} = wzD_q + zD_x,\ e_{15} = D_z - \frac{wzD_q}{2},\\
e_{16} = (zw - 2y)D_t,\ e_{17} = (\frac{-wz}{2} + y)D_q,\ e_{18} = \frac{w^3}{3}D_q + \frac{w^2}{2}D_x + wD_y,\\
e_{19} = (w^2z - 2wy + 4x)D_q + zwD_x + 2zD_y,\\
e_{20} = (\frac{-zw^2}{2} + yw + \frac{w^4}{12} - 2x)D_q + \frac{w^3D_x}{6} + \frac{w^2D_y}{2} + wD_z,\ e_{21} = tD_t,\\
e_{22} = (\frac{zw^3}{3} - yw^2 + 2xw + q)D_q + wD_w + (\frac{w^2z}{2} - wy + 2x)D_x + (wz - y)D_y,\\
e_{23} = (-\frac{zw^3}{6} + \frac{yw^2}{2} - xw + 2q)D_q + (-\frac{w^2z}{4} + \frac{wy}{2} + x)D_x + (2y - \frac{wz}{2})D_y + zD_z.
\end{aligned}$$

and their nonvanishing Lie brackets are

$$
\begin{array}{lll}
[e_1, e_4] = e_5, & [e_1, e_{11}] = e_2 + e_8, & [e_1, e_{21}] = e_1, \\
[e_2, e_{13}] = 2e_5, & [e_2, e_{19}] = 4e_5, & [e_2, e_{20}] = -2e_5, \\
[e_2, e_{22}] = 2e_2 + 2e_8, & [e_2, e_{23}] = e_2 - e_8, & [e_3, e_{13}] = e_2, \\
[e_3, e_{16}] = -2e_1, & [e_3, e_{17}] = e_5, & [e_3, e_{19}] = -2e_8, \\
[e_3, e_{20}] = e_8, & [e_3, e_{22}] = -e_{12} - e_3, & [e_3, e_{23}] = \dfrac{e_{12}}{2} + 2e_3, \\
[e_4, e_9] = -e_8, & [e_4, e_{10}] = -e_7, & [e_4, e_{16}] = 2e_{17}, \\
[e_4, e_{21}] = -e_4, & [e_4, e_{22}] = e_4, & [e_4, e_{23}] = 2e_4, \\
[e_5, e_{22}] = e_5, & [e_5, e_{23}] = 2e_5, & [e_6, e_8] = e_5, \\
[e_6, e_9] = e_1, & [e_6, e_{11}] = e_4, & [e_6, e_{12}] = e_2 + 2e_8, \\
[e_6, e_{14}] = e_7, & [e_6, e_{15}] = -\dfrac{e_7}{2}, & [e_6, e_{16}] = e_{10}, \\
[e_6, e_{17}] = -\dfrac{e_7}{2}, & [e_6, e_{18}] = e_{12} + e_3, & [e_6, e_{19}] = e_{14} - 2e_{17}, \\
[e_6, e_{20}] = e_{18} + e_{15} + e_{17}, & [e_6, e_{22}] = e_{19} - e_{13} + e_6, & [e_6, e_{23}] = \dfrac{e_{13}}{2} - \dfrac{e_{19}}{2}, \\
[e_7, e_{15}] = -e_5, & [e_7, e_{20}] = -e_8, & [e_7, e_{22}] = e_7, \\
[e_7, e_{23}] = e_7, & [e_8, e_{23}] = 2e_8, & [e_9, e_{11}] = e_{12}, \\
[e_9, e_{21}] = e_9, & [e_9, e_{22}] = -e_9, & [e_{10}, e_{11}] = e_{14}, \\
[e_{10}, e_{15}] = -e_1, & [e_{10}, e_{20}] = -e_9, & [e_{10}, e_{21}] = e_{10}, \\
[e_{10}, e_{23}] = -e_{10}, & [e_{11}, e_{13}] = 2e_4, & [e_{11}, e_{16}] = 2e_{13} - e_{19}, \\
[e_{11}, e_{19}] = 4e_4, & [e_{11}, e_{20}] = -2e_4, & [e_{11}, e_{21}] = -e_{11}, \\
[e_{11}, e_{22}] = 2e_{11}, & [e_{11}, e_{23}] = e_{11}, & [e_{12}, e_{13}] = 2e_8, \\
[e_{12}, e_{19}] = 4e_8, & [e_{12}, e_{20}] = -2e_8, & [e_{12}, e_{22}] = e_{12}, \\
[e_{12}, e_{23}] = e_{12}, & [e_{13}, e_{14}] = -2e_7, & [e_{13}, e_{15}] = -e_3,
\end{array}
$$

$$[e_{13}, e_{16}] = -2e_{10}, \quad [e_{13}, e_{17}] = e_7, \quad [e_{13}, e_{18}] = -e_{12},$$

$$[e_{13}, e_{19}] = -2e_{14} + 4e_{17}, \quad [e_{13}, e_{20}] = -2e_{17} - e_{18}, \quad [e_{13}, e_{22}] = 3e_{13} - 2e_{19},$$

$$[e_{13}, e_{23}] = -e_{13} + e_{19}, \quad [e_{14}, e_{15}] = -e_2 - e_8, \quad [e_{14}, e_{19}] = 4e_7,$$

$$[e_{14}, e_{20}] = -e_{12} - 2e_7, \quad [e_{14}, e_{22}] = 2e_{14}, \quad [e_{15}, e_{16}] = e_9,$$

$$[e_{15}, e_{17}] = -\frac{e_8}{2}, \quad [e_{15}, e_{19}] = e_{12} + 2e_3, \quad [e_{15}, e_{22}] = e_{18},$$

$$[e_{15}, e_{23}] = e_{15} - \frac{e_{18}}{2}, \quad [e_{16}, e_{18}] = 2e_9, \quad [e_{16}, e_{19}] = 4e_{10},$$

$$[e_{16}, e_{21}] = e_{16}, \quad [e_{16}, e_{22}] = e_{16}, \quad [e_{16}, e_{23}] = -2e_{16},$$

$$[e_{17}, e_{18}] = -e_8, \quad [e_{17}, e_{19}] = -2e_7, \quad [e_{17}, e_{22}] = 2e_{17},$$

$$[e_{18}, e_{22}] = -2e_{18}, \quad [e_{18}, e_{23}] = 2e_{18}, \quad [e_{19}, e_{20}] = -2e_{18},$$

$$[e_{19}, e_{22}] = -e_{19}, \quad [e_{19}, e_{23}] = e_{19}, \quad [e_{20}, e_{22}] = -e_{20},$$

$$[e_{20}, e_{23}] = e_{20}.$$

We conclude that the symmetry algebra is a twenty-three-dimensional solvable, where the nilradical is a twenty-dimensional spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}$ and an abelian complement spanned by e_{21}, e_{22}, e_{23}.

CHAPTER 4

CLASSIFICATION OF THE SYMMETRY LIE ALGEBRAS OF THE CANONICAL GEODESIC EQUATIONS OF FIVE-DIMENSIONAL SOLVABLE LIE ALGEBRAS

The focus of this chapter is to construct and classify the symmetry algebras of geodesics, [47], associated with the five-dimensional solvable Lie algebras [35]. In particular, we consider the geodesics that correspond to the algebras $A_{5,7}^{abc}$ through $A_{5,18}^{a}$. We dedicate a separate section to each of the twelve class of such algebras and its corresponding geodesic systems. Ten of the geodesic equations contain parameters as their original corresponding algebras involve some essential parameters; therefore, there is always a certain amount of arbitrariness to be considered. Consequently, we devote a separate subsection to each subcase.

For each case, we methodically provide the nonzero brackets of the original Lie algebra, the associated system of geodesics, a basis for the associated Lie algebra of symmetries, and the corresponding nonvanishing Lie brackets. We discuss each case in turn and draw various conclusions about their symmetry algebra properties. It should be pointed out that determining the symmetry algebra basis and identifying its Lie algebraic structure in each of these cases is facilitated and verified by the MAPLE symbolic manipulation program explained in Section 2.6.

We remark that, for exceptional values of the parameters, the dimension of symmetry algebras appear to be larger than that of other comparable algebras. There may be two underlying causes that help to describe this phenomenon. First, solvable algebras, at least in dimension five, depend on parameters. Furthermore, it seems as though geodesic systems for *some* solvable Lie algebras do not always contain more than one trivial geodesic equation, that is, when the right-hand side is zero. Finally, we observe that the complement of the radical of case ten is not a Lie algebra, and this may be because its geodesics are trivial and did not contain parameters.

4.1 Algebra$_{5,7}^{abc}$

We have already examined the symmetry algebra of the geodesics for $A_{5,7}^{abc}$ in Section 2.6, as an illumination of the efficiency of our algorithmic computation scheme proposed in this dissertation. Nevertheless, we iterate the case and result concluded there in this section for completeness. The nonzero brackets of $A_{5,7}^{abc}$ are

$$[e_1, e_5] = e_1,\ [e_2, e_5] = ae_2,\ [e_3, e_5] = be_3,\ [e_4, e_5] = ce_4;\ (abc \neq 0, -1 \leq c \leq b \leq a \leq 1), \tag{4.1}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = a\dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w}, \quad \ddot{z} = c\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.2}$$

The Lie algebra of symmetries of (4.2) is a sixteen-dimensional spanned by the basis

$$\begin{array}{llllll} e_1 = D_z, & e_2 = e^{bw}D_y, & e_3 = e^{aw}D_x, & e_4 = D_t, & e_5 = D_q, & e_6 = D_x, \\ e_7 = D_y, & e_8 = e^{cw}D_z, & e_9 = e^{w}D_q, & e_{10} = wD_t, & e_{11} = xD_x, & e_{12} = yD_y, \\ e_{13} = zD_z, & e_{14} = qD_q, & e_{15} = tD_t, & e_{16} = D_w. & & \end{array}$$

Their nonvanishing brackets are

$$\begin{array}{lllll} [e_1, e_{13}] = e_1, & [e_2, e_{12}] = e_2, & [e_2, e_{16}] = -be_2, & [e_3, e_{11}] = e_3, & [e_3, e_{16}] = -ae_3, \\ [e_4, e_{15}] = e_4, & [e_5, e_{14}] = e_5, & [e_6, e_{11}] = e_6, & [e_7, e_{12}] = e_7, & [e_8, e_{13}] = e_8, \\ [e_8, e_{16}] = -ce_8, & [e_9, e_{14}] = e_9, & [e_9, e_{16}] = -e_9, & [e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = -e_4. \end{array}$$

For such a generic case, the symmetry algebra is a sixteen-dimensional indecomposable solvable. It has a ten-dimensional abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6$, e_7, e_8, e_9, e_{10} and a six-dimensional abelian complement spanned by $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$, e_{16}. The symmetry algebra as a whole is isomorphic to $\mathbb{R}^6 \rtimes \mathbb{R}^{10}$.

4.1.1 Subcase $a = 1, bc \neq 1$

The symmetries and nonzero Lie brackets are, respectively,

$$\begin{array}{lllll}
e_1 = D_x, & e_2 = D_t, & e_3 = D_y, & e_4 = D_z, & e_5 = D_q, \\
e_6 = wD_t, & e_7 = e^w D_q, & e_8 = e^w D_x, & e_9 = e^{bw} D_y, & e_{10} = e^{cw} D_z, \\
e_{11} = D_w, & e_{12} = tD_t, & e_{13} = yD_y, & e_{14} = qD_q + xD_x, & e_{15} = zD_z, \\
e_{16} = qD_x, & e_{17} = -qD_q + xD_x, & e_{18} = xD_q. & &
\end{array}$$

$$\begin{array}{lllll}
[e_1, e_{14}] = e_1, & [e_1, e_{17}] = e_1, & [e_1, e_{18}] = e_5, & [e_2, e_{12}] = e_2, & [e_3, e_{13}] = e_3, \\
[e_4, e_{15}] = e_4, & [e_5, e_{14}] = e_5, & [e_5, e_{16}] = e_1, & [e_5, e_{17}] = -e_5, & [e_6, e_{11}] = -e_2, \\
[e_6, e_{12}] = e_6, & [e_7, e_{11}] = -e_7, & [e_7, e_{14}] = e_7, & [e_7, e_{16}] = e_8, & [e_7, e_{17}] = -e_7, \\
[e_8, e_{11}] = -e_8, & [e_8, e_{14}] = e_8, & [e_8, e_{17}] = e_8, & [e_8, e_{18}] = e_7, & [e_9, e_{11}] = -be_9, \\
[e_9, e_{13}] = e_9, & [e_{10}, e_{11}] = -ce_{10}, & [e_{10}, e_{15}] = e_{10}, & [e_{16}, e_{17}] = 2e_{16}, & [e_{16}, e_{18}] = -e_{17}, \\
[e_{17}, e_{18}] = 2e_{18}. & & & &
\end{array}$$

4.1.2 Subcase $a = b, a \neq c$

The symmetries and nonzero Lie brackets are, respectively,

$$\begin{array}{lllll}
e_1 = D_y, & e_2 = D_t, & e_3 = D_q, & e_4 = D_z, & e_5 = D_x, \\
e_6 = wD_t, & e_7 = e^w D_q, & e_8 = e^{bw} D_x, & e_9 = e^{bw} D_y, & e_{10} = e^{cw} D_z, \\
e_{11} = zD_z, & e_{12} = D_w, & e_{13} = tD_t, & e_{14} = qD_q, & e_{15} = xD_x + yD_y,
\end{array}$$

$$e_{16} = xD_y, \quad e_{17} = -xD_x + yD_y, \quad e_{18} = yD_x.$$

$$[e_1, e_{15}] = e_1, \quad [e_1, e_{17}] = e_1, \quad [e_1, e_{18}] = e_5, \quad [e_2, e_{13}] = e_2, \quad [e_3, e_{14}] = e_3,$$
$$[e_4, e_{11}] = e_4, \quad [e_5, e_{15}] = e_5, \quad [e_5, e_{16}] = e_1, \quad [e_5, e_{17}] = -e_5, \quad [e_6, e_{12}] = -e_2,$$
$$[e_6, e_{13}] = e_6, \quad [e_7, e_{12}] = -e_7, \quad [e_7, e_{14}] = e_7, \quad [e_8, e_{12}] = -be_8, \quad [e_8, e_{15}] = e_8,$$
$$[e_8, e_{16}] = e_9, \quad [e_8, e_{17}] = -e_8, \quad [e_9, e_{12}] = -be_9, \quad [e_9, e_{15}] = e_9, \quad [e_9, e_{17}] = e_9,$$
$$[e_9, e_{18}] = e_8, \quad [e_{10}, e_{11}] = e_{10}, \quad [e_{10}, e_{12}] = -ce_{10}, \quad [e_{16}, e_{17}] = 2e_{16}, \quad [e_{16}, e_{18}] = -e_{17},$$
$$[e_{17}, e_{18}] = 2e_{18}.$$

4.1.3 Subcase $b = c, a \neq bc$

The symmetries and nonzero Lie brackets are, respectively,

$$e_1 = D_z, \quad e_2 = D_t, \quad e_3 = D_q, \quad e_4 = D_x, \quad e_5 = D_y,$$
$$e_6 = e^{aw}D_x, \quad e_7 = wD_t, \quad e_8 = e^{cw}D_z, \quad e_9 = e^{w}D_q, \quad e_{10} = e^{cw}D_y,$$
$$e_{11} = tD_t, \quad e_{12} = xD_x, \quad e_{13} = qD_q, \quad e_{14} = D_w, \quad e_{15} = yD_y + zD_z,$$
$$e_{16} = yD_z, \quad e_{17} = -yD_y + zD_z, \quad e_{18} = zD_y.$$

$$[e_1, e_{15}] = e_1, \quad [e_1, e_{17}] = e_1, \quad [e_1, e_{18}] = e_5, \quad [e_2, e_{11}] = e_2, \quad [e_3, e_{13}] = e_3,$$
$$[e_4, e_{12}] = e_4, \quad [e_5, e_{15}] = e_5, \quad [e_5, e_{16}] = e_1, \quad [e_5, e_{17}] = -e_5, \quad [e_6, e_{12}] = e_6,$$
$$[e_6, e_{14}] = -ae_6, \quad [e_7, e_{11}] = e_7, \quad [e_7, e_{14}] = -e_2, \quad [e_8, e_{14}] = -ce_8, \quad [e_8, e_{15}] = e_8,$$
$$[e_8, e_{17}] = e_8, \quad [e_8, e_{18}] = e_{10}, \quad [e_9, e_{13}] = e_9, \quad [e_9, e_{14}] = -e_9, \quad [e_{10}, e_{14}] = -ce_{10},$$
$$[e_{10}, e_{15}] = e_{10}, \quad [e_{10}, e_{16}] = e_8, \quad [e_{10}, e_{17}] = -e_{10}, \quad [e_{16}, e_{17}] = 2e_{16}, \quad [e_{16}, e_{18}] = -e_{17},$$
$$[e_{17}, e_{18}] = 2e_{18}.$$

For the three subcases above, the Lie symmetry algebra for each case is an indecomposable Levi decomposition $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^5 \rtimes \mathbb{R}^{10})$, where the semisimple part is spanned by e_{16}, e_{17}, e_{18}. The radical consists of a ten-dimensional indecomposable nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and a five-dimensional abelian complement spanned by $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$.

4.1.4 Subcase $a = 1, b = 1, c \neq 1$

The symmetries and nonzero Lie brackets are, respectively,

$$\begin{array}{lllll}
e_1 = D_x, & e_2 = D_y, & e_3 = D_z, & e_4 = D_t, & e_5 = D_q, \\
e_6 = e^w D_y, & e_7 = e^w D_q, & e_8 = e^{cw} D_z, & e_9 = wD_t, & e_{10} = e^w D_x, \\
e_{11} = D_w, & e_{12} = tD_t, & e_{13} = zD_z, & e_{14} = qD_q + xD_x + yD_y, & e_{15} = yD_x, \\
e_{16} = qD_y, & e_{17} = xD_y, & e_{18} = -qD_q + xD_x, & e_{19} = -qD_q + yD_y, & e_{20} = xD_q, \\
e_{21} = yD_q, & e_{22} = qD_x. & & &
\end{array}$$

$$\begin{array}{llll}
[e_1, e_{14}] = e_1, & [e_1, e_{17}] = e_2, & [e_1, e_{18}] = e_1, & [e_1, e_{20}] = e_5, \\
[e_2, e_{14}] = e_2, & [e_2, e_{15}] = e_1, & [e_2, e_{19}] = e_2, & [e_2, e_{21}] = e_5, \\
[e_3, e_{13}] = e_3, & [e_4, e_{12}] = e_4, & [e_5, e_{14}] = e_5, & [e_5, e_{16}] = e_2, \\
[e_5, e_{18}] = -e_5, & [e_5, e_{19}] = -e_5, & [e_5, e_{22}] = e_1, & [e_6, e_{11}] = -e_6, \\
[e_6, e_{14}] = e_6, & [e_6, e_{15}] = e_{10}, & [e_6, e_{19}] = e_6, & [e_6, e_{21}] = e_7, \\
[e_7, e_{11}] = -e_7, & [e_7, e_{14}] = e_7, & [e_7, e_{16}] = e_6, & [e_7, e_{18}] = -e_7, \\
[e_7, e_{19}] = -e_7, & [e_7, e_{22}] = e_{10}, & [e_8, e_{11}] = -ce_8, & [e_8, e_{13}] = e_8, \\
[e_9, e_{11}] = -e_4, & [e_9, e_{12}] = e_9, & [e_{10}, e_{11}] = -e_{10}, & [e_{10}, e_{14}] = e_{10}, \\
[e_{10}, e_{17}] = e_6, & [e_{10}, e_{18}] = e_{10}, & [e_{10}, e_{20}] = e_7, & [e_{15}, e_{16}] = -e_{22}, \\
[e_{15}, e_{17}] = -e_{18} + e_{19}, & [e_{15}, e_{18}] = e_{15}, & [e_{15}, e_{19}] = -e_{15}, & [e_{15}, e_{20}] = e_{21},
\end{array}$$

$$[e_{16}, e_{18}] = e_{16}, \quad [e_{16}, e_{19}] = 2e_{16}, \quad [e_{16}, e_{20}] = -e_{17}, \quad [e_{16}, e_{21}] = -e_{19},$$
$$[e_{17}, e_{18}] = -e_{17}, \quad [e_{17}, e_{19}] = e_{17}, \quad [e_{17}, e_{21}] = e_{20}, \quad [e_{17}, e_{22}] = -e_{16},$$
$$[e_{18}, e_{20}] = 2e_{20}, \quad [e_{18}, e_{21}] = e_{21}, \quad [e_{18}, e_{22}] = -2e_{22}, \quad [e_{19}, e_{20}] = e_{20},$$
$$[e_{19}, e_{21}] = 2e_{21}, \quad [e_{19}, e_{22}] = -e_{22}, \quad [e_{20}, e_{22}] = e_{18}, \quad [e_{21}, e_{22}] = e_{15}.$$

4.1.5 Subcase $a = b = c$

The symmetries and nonzero Lie brackets are, respectively,

$$e_1 = D_y, \quad e_2 = D_z, \quad e_3 = D_q, \quad e_4 = D_t,$$
$$e_5 = D_x, \quad e_6 = e^{cw} D_y, \quad e_7 = e^{cw} D_z, \quad e_8 = e^{cw} D_x,$$
$$e_9 = wD_t, \quad e_{10} = e^w D_q, \quad e_{11} = tD_t, \quad e_{12} = qD_q,$$
$$e_{13} = D_w, \quad e_{14} = xD_x + yD_y + zD_z, \quad e_{15} = zD_y, \quad e_{16} = xD_z,$$
$$e_{17} = yD_z, \quad e_{18} = -xD_x + yD_y, \quad e_{19} = -xD_x + zD_z, \quad e_{20} = yD_x,$$
$$e_{21} = zD_x, \quad e_{22} = xD_y$$

$$[e_1, e_{14}] = e_1, \quad [e_1, e_{17}] = e_2, \quad [e_1, e_{18}] = e_1, \quad [e_1, e_{20}] = e_5,$$
$$[e_2, e_{14}] = e_2, \quad [e_2, e_{15}] = e_1, \quad [e_2, e_{19}] = e_2, \quad [e_2, e_{21}] = e_5,$$
$$[e_3, e_{12}] = e_3, \quad [e_4, e_{11}] = e_4, \quad [e_5, e_{14}] = e_5, \quad [e_5, e_{16}] = e_2,$$
$$[e_5, e_{18}] = -e_5, \quad [e_5, e_{19}] = -e_5, \quad [e_5, e_{22}] = e_1, \quad [e_6, e_{13}] = -ce_6,$$
$$[e_6, e_{14}] = e_6, \quad [e_6, e_{17}] = e_7, \quad [e_6, e_{18}] = e_6, \quad [e_6, e_{20}] = e_8,$$
$$[e_7, e_{13}] = -ce_7, \quad [e_7, e_{14}] = e_7, \quad [e_7, e_{15}] = e_6, \quad [e_7, e_{19}] = e_7,$$
$$[e_7, e_{21}] = e_8, \quad [e_8, e_{13}] = -ce_8, \quad [e_8, e_{14}] = e_8, \quad [e_8, e_{16}] = e_7,$$
$$[e_8, e_{18}] = -e_8, \quad [e_8, e_{19}] = -e_8, \quad [e_8, e_{22}] = e_6, \quad [e_9, e_{11}] = e_9,$$
$$[e_9, e_{13}] = -e_4, \quad [e_{10}, e_{12}] = e_{10}, \quad [e_{10}, e_{13}] = -e_{10}, \quad [e_{15}, e_{16}] = -e_{22},$$

$$
\begin{array}{llll}
[e_{15}, e_{17}] = -e_{18} + e_{19}, & [e_{15}, e_{18}] = e_{15}, & [e_{15}, e_{19}] = -e_{15}, & [e_{15}, e_{20}] = e_{21}, \\
[e_{16}, e_{18}] = e_{16}, & [e_{16}, e_{19}] = 2e_{16}, & [e_{16}, e_{20}] = -e_{17}, & [e_{16}, e_{21}] = -e_{19}, \\
[e_{17}, e_{18}] = -e_{17}, & [e_{17}, e_{19}] = e_{17}, & [e_{17}, e_{21}] = e_{20}, & [e_{17}, e_{22}] = -e_{16}, \\
[e_{18}, e_{20}] = 2e_{20}, & [e_{18}, e_{21}] = e_{21}, & [e_{18}, e_{22}] = -2e_{22}, & [e_{19}, e_{20}] = e_{20}, \\
[e_{19}, e_{21}] = 2e_{21}, & [e_{19}, e_{22}] = -e_{22}, & [e_{20}, e_{22}] = e_{18}, & [e_{21}, e_{22}] = e_{15}.
\end{array}
$$

For both subcases, the symmetry algebra is a twenty-two-dimensional indecomposable Levi decomposition with an eight-dimensional semisimple $\mathfrak{sl}(3,\mathbb{R})$ spanned by $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$ as well as a fourteen-dimensional solvable consisting of a ten-dimensional abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and a four-dimensional abelian complement spanned by $e_{11}, e_{12}, e_{13}, e_{14}$.

4.1.6 Subcase $a = 1, b = 1, c = 1$

The symmetries and nonzero Lie brackets are, respectively,

$$
\begin{array}{llll}
e_1 = D_x, & e_2 = D_y, & e_3 = D_z, & e_4 = D_t, \\
e_5 = D_q, & e_6 = e^w D_x, & e_7 = e^w D_y, & e_8 = e^w D_z, \\
e_9 = e^w D_q, & e_{10} = wD_t, & e_{11} = tD_t, & e_{12} = qD_q + xD_x + yD_y + zD_z, \\
e_{13} = D_w, & e_{14} = zD_x, & e_{15} = qD_y, & e_{16} = xD_y, \\
e_{17} = zD_y, & e_{18} = qD_q - xD_x, & e_{19} = -xD_x + yD_y, & e_{20} = qD_z, \\
e_{21} = xD_q, & e_{22} = xD_z, & e_{23} = yD_z, & e_{24} = -xD_x + zD_z, \\
e_{25} = yD_q, & e_{26} = zD_q, & e_{27} = qD_x, & e_{28} = yD_x.
\end{array}
$$

$$
\begin{array}{llll}
[e_1, e_{12}] = e_1, & [e_1, e_{16}] = e_2, & [e_1, e_{18}] = -e_1, & [e_1, e_{19}] = -e_1, \\
[e_1, e_{21}] = e_5, & [e_1, e_{22}] = e_3, & [e_1, e_{24}] = -e_1, & [e_2, e_{12}] = e_2,
\end{array}
$$

$$
\begin{array}{llll}
[e_2, e_{19}] = e_2, & [e_2, e_{23}] = e_3, & [e_2, e_{25}] = e_5, & [e_2, e_{28}] = e_1, \\
[e_3, e_{12}] = e_3, & [e_3, e_{14}] = e_1, & [e_3, e_{17}] = e_2, & [e_3, e_{24}] = e_3, \\
[e_3, e_{26}] = e_5, & [e_4, e_{11}] = e_4, & [e_5, e_{12}] = e_5, & [e_5, e_{15}] = e_2, \\
[e_5, e_{18}] = e_5, & [e_5, e_{20}] = e_3, & [e_5, e_{27}] = e_1, & [e_6, e_{12}] = e_6, \\
[e_6, e_{13}] = -e_6, & [e_6, e_{16}] = e_7, & [e_6, e_{18}] = -e_6, & [e_6, e_{19}] = -e_6, \\
[e_6, e_{21}] = e_9, & [e_6, e_{22}] = e_8, & [e_6, e_{24}] = -e_6, & [e_7, e_{12}] = e_7, \\
[e_7, e_{13}] = -e_7, & [e_7, e_{19}] = e_7, & [e_7, e_{23}] = e_8, & [e_7, e_{25}] = e_9, \\
[e_7, e_{28}] = e_6, & [e_8, e_{12}] = e_8, & [e_8, e_{13}] = -e_8, & [e_8, e_{14}] = e_6, \\
[e_8, e_{17}] = e_7, & [e_8, e_{24}] = e_8, & [e_8, e_{26}] = e_9, & [e_9, e_{12}] = e_9, \\
[e_9, e_{13}] = -e_9, & [e_9, e_{15}] = e_7, & [e_9, e_{18}] = e_9, & [e_9, e_{20}] = e_8, \\
[e_9, e_{27}] = e_6, & [e_{10}, e_{11}] = e_{10}, & [e_{10}, e_{13}] = -e_4, & [e_{14}, e_{16}] = e_{17}, \\
[e_{14}, e_{18}] = -e_{14}, & [e_{14}, e_{19}] = -e_{14}, & [e_{14}, e_{20}] = -e_{27}, & [e_{14}, e_{21}] = e_{26}, \\
[e_{14}, e_{22}] = e_{24}, & [e_{14}, e_{23}] = -e_{28}, & [e_{14}, e_{24}] = -2e_{14}, & [e_{15}, e_{18}] = -e_{15}, \\
[e_{15}, e_{19}] = e_{15}, & [e_{15}, e_{21}] = -e_{16}, & [e_{15}, e_{23}] = e_{20}, & [e_{15}, e_{25}] = e_{18} - e_{19}, \\
[e_{15}, e_{26}] = -e_{17}, & [e_{15}, e_{28}] = e_{27}, & [e_{16}, e_{18}] = e_{16}, & [e_{16}, e_{19}] = 2e_{16}, \\
[e_{16}, e_{23}] = e_{22}, & [e_{16}, e_{24}] = e_{16}, & [e_{16}, e_{25}] = e_{21}, & [e_{16}, e_{27}] = -e_{15}, \\
[e_{16}, e_{28}] = -e_{19}, & [e_{17}, e_{19}] = e_{17}, & [e_{17}, e_{20}] = -e_{15}, & [e_{17}, e_{22}] = -e_{16}, \\
[e_{17}, e_{23}] = -e_{19} + e_{24}, & [e_{17}, e_{24}] = -e_{17}, & [e_{17}, e_{25}] = e_{26}, & [e_{17}, e_{28}] = e_{14}, \\
[e_{18}, e_{20}] = e_{20}, & [e_{18}, e_{21}] = -2e_{21}, & [e_{18}, e_{22}] = -e_{22}, & [e_{18}, e_{25}] = -e_{25}, \\
[e_{18}, e_{26}] = -e_{26}, & [e_{18}, e_{27}] = 2e_{27}, & [e_{18}, e_{28}] = e_{28}, & [e_{19}, e_{21}] = -e_{21}, \\
[e_{19}, e_{22}] = -e_{22}, & [e_{19}, e_{23}] = e_{23}, & [e_{19}, e_{25}] = e_{25}, & [e_{19}, e_{27}] = e_{27}, \\
[e_{19}, e_{28}] = 2e_{28}, & [e_{20}, e_{21}] = -e_{22}, & [e_{20}, e_{24}] = e_{20}, & [e_{20}, e_{25}] = -e_{23}, \\
[e_{20}, e_{26}] = e_{18} - e_{24}, & [e_{21}, e_{24}] = e_{21}, & [e_{21}, e_{27}] = -e_{18}, & [e_{21}, e_{28}] = -e_{25},
\end{array}
$$

$$[e_{22}, e_{24}] = 2e_{22}, \quad [e_{22}, e_{26}] = e_{21}, \quad [e_{22}, e_{27}] = -e_{20}, \quad [e_{22}, e_{28}] = -e_{23},$$
$$[e_{23}, e_{24}] = e_{23}, \quad [e_{23}, e_{26}] = e_{25}, \quad [e_{24}, e_{26}] = e_{26}, \quad [e_{24}, e_{27}] = e_{27},$$
$$[e_{24}, e_{28}] = e_{28}, \quad [e_{25}, e_{27}] = e_{28}, \quad [e_{26}, e_{27}] = e_{14}.$$

It is a twenty-eight-dimensional indecomposable with nontrivial Levi decomposition $\mathfrak{sl}(4, \mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathbb{R}^{10})$. The semisimple is spanned by $e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}, e_{23}, e_{24}, e_{25}, e_{26}, e_{27}, e_{28}$. The radical is a semi direct product of a ten-dimensional indecomposable nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and a three-dimensional abelian complement spanned by e_{11}, e_{12}, e_{13}.

4.2 Algebra$^c_{5,8}$

The nonzero brackets are

$$[e_2, e_5] = e_1,\ [e_3, e_5] = e_3,\ [e_4, e_5] = ce_4; \quad (0 \le |c| \le 1), \tag{4.3}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{x}\dot{w}, \quad \ddot{x} = 0, \quad \ddot{y} = \dot{y}\dot{w}, \quad \ddot{z} = c\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.4}$$

The symmetry algebra basis and nonvanishing brackets are, respectively,

$$e_1 = D_x, \quad e_2 = D_t, \quad e_3 = D_y,$$
$$e_4 = D_q, \quad e_5 = D_z, \quad e_6 = wD_t,$$
$$e_7 = wD_q, \quad e_8 = e^w D_y, \quad e_9 = e^{cw} D_t,$$
$$e_{10} = \frac{1}{2}w^2 D_q + wD_x, \quad e_{11} = \frac{xD_q}{2} + D_w, \quad e_{12} = qD_q + tD_t + xD_x,$$
$$e_{13} = yD_y, \quad e_{14} = zD_z, \quad e_{15} = \frac{(wx - 2q)D_q}{2} + tD_t,$$
$$e_{16} = tD_q, \quad e_{17} = xD_t, \quad e_{18} = xD_q,$$

$$e_{19} = qD_q + xD_x + \big(xw - 2q\big)D_q, \qquad e_{20} = tD_x + \frac{twD_q}{2}, \quad e_{21} = \big(xw - 2q\big)D_t,$$

$$e_{22} = \big(\frac{xw^2}{2} - qw\big)D_q + \big(xw - 2q\big)D_x.$$

$$\begin{array}{llll}
[e_1, e_{11}] = \frac{e_4}{2}, & [e_1, e_{12}] = e_1, & [e_1, e_{15}] = \frac{e_7}{2}, & [e_1, e_{17}] = e_2, \\
[e_1, e_{18}] = e_4, & [e_1, e_{19}] = e_1 + e_7, & [e_1, e_{21}] = e_6, & [e_1, e_{22}] = e_{10}, \\
[e_2, e_{12}] = e_2, & [e_2, e_{15}] = e_2, & [e_2, e_{16}] = e_4, & [e_2, e_{20}] = e_1 + \frac{e_7}{2}, \\
[e_3, e_{13}] = e_3, & [e_4, e_{12}] = e_4, & [e_4, e_{15}] = -e_4, & [e_4, e_{19}] = -e_4, \\
[e_4, e_{21}] = -2e_2, & [e_4, e_{22}] = -2e_1 - e_7, & [e_5, e_{14}] = e_5, & [e_6, e_{11}] = -e_2, \\
[e_6, e_{12}] = e_6, & [e_6, e_{15}] = e_6, & [e_6, e_{16}] = e_7, & [e_6, e_{20}] = -e_{10}, \\
[e_7, e_{11}] = -e_4, & [e_7, e_{12}] = e_7, & [e_7, e_{15}] = -e_7, & [e_7, e_{19}] = -e_7, \\
[e_7, e_{21}] = -2e_6, & [e_7, e_{22}] = -2e_{10}, & [e_8, e_{11}] = -e_8, & [e_8, e_{13}] = e_8, \\
[e_9, e_{11}] = -ce_9, & [e_9, e_{14}] = e_9, & [e_{10}, e_{11}] = -e_1 - \frac{e_7}{2}, & [e_{10}, e_{12}] = e_{10}, \\
[e_{10}, e_{17}] = e_6, & [e_{10}, e_{18}] = e_7, & [e_{10}, e_{19}] = e_{10}, & [e_{15}, e_{16}] = 2e_{16}, \\
[e_{15}, e_{17}] = -e_{17}, & [e_{15}, e_{18}] = e_{18}, & [e_{15}, e_{20}] = e_{20}, & [e_{15}, e_{21}] = -2e_{21}, \\
[e_{15}, e_{22}] = -e_{22}, & [e_{16}, e_{17}] = -e_{18}, & [e_{16}, e_{19}] = -e_{16}, & [e_{16}, e_{21}] = -2e_{15}, \\
[e_{16}, e_{22}] = -2e_{20}, & [e_{17}, e_{19}] = -e_{17}, & [e_{17}, e_{20}] = -e_{15} + e_{19}, & [e_{17}, e_{22}] = -e_{21}, \\
[e_{18}, e_{19}] = -2e_{18}, & [e_{18}, e_{20}] = -e_{16}, & [e_{18}, e_{21}] = -2e_{17}, & [e_{18}, e_{22}] = -2e_{19}, \\
[e_{19}, e_{20}] = -e_{20}, & [e_{19}, e_{21}] = -e_{21}, & [e_{19}, e_{22}] = -2e_{22}, & [e_{20}, e_{21}] = -e_{22}.
\end{array}$$

For the generic case, the symmetry algebra is $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R}^4 \rtimes \mathbb{R}^{10})$. The semisimple part $\mathfrak{sl}(3,\mathbb{R})$ is spanned by $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$; the abelian nilradical $\mathbb{R}^{10}$ is spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$, and the abelian complement to $\mathbb{R}^{10}$ is spanned by $e_{11}, e_{12}, e_{13}, e_{14}$.

4.2.1 Subcase $c = 1$

The symmetries and nonzero brackets are, respectively,

$$
\begin{aligned}
&e_1 = D_x, && e_2 = D_t,\\
&e_3 = D_q, && e_4 = D_y,\\
&e_5 = D_z, && e_6 = wD_t,\\
&e_7 = wD_q, && e_8 = e^w D_y,\\
&e_9 = e^w D_z, && e_{10} = \frac{w^2}{2}D_q + wD_x,\\
&e_{11} = D_w + \frac{x}{2}D_q, && e_{12} = qD_q + tD_t + xD_x,\\
&e_{13} = yD_y + zD_z, && e_{14} = yD_y - zD_z,\\
&e_{15} = zD_y, && e_{16} = yD_z,\\
&e_{17} = tD_t + \frac{(wx-2q)}{2}D_q, && e_{18} = xD_q, \quad e_{19} = tD_q,\\
&e_{20} = qD_q + xD_x + (wx-2q)D_q, && e_{21} = xD_t,\\
&e_{22} = \frac{tw}{2}D_q + tD_x, && e_{23} = (wx-2q)D_t,\\
&e_{24} = \frac{(xw^2-qw)}{2}D_q + (wx-2q)D_x.
\end{aligned}
$$

$$
\begin{aligned}
&[e_1, e_{11}] = \frac{e_3}{2}, && [e_1, e_{12}] = e_1, && [e_1, e_{17}] = \frac{e_7}{2}, && [e_1, e_{18}] = e_3,\\
&[e_1, e_{20}] = e_1 + e_7, && [e_1, e_{21}] = e_2, && [e_1, e_{23}] = e_6, && [e_1, e_{24}] = e_{10},\\
&[e_2, e_{12}] = e_2, && [e_2, e_{17}] = e_2, && [e_2, e_{19}] = e_3, && [e_2, e_{22}] = e_1 + \frac{e_7}{2},\\
&[e_3, e_{12}] = e_3, && [e_3, e_{17}] = -e_3, && [e_3, e_{20}] = -e_3, && [e_3, e_{23}] = -2e_2,\\
&[e_3, e_{24}] = -2e_1 - e_7, && [e_4, e_{13}] = e_4, && [e_4, e_{14}] = e_4, && [e_4, e_{16}] = e_5,\\
&[e_5, e_{13}] = e_5, && [e_5, e_{14}] = -e_5, && [e_5, e_{15}] = e_4, && [e_6, e_{11}] = -e_2,\\
&[e_6, e_{12}] = e_6, && [e_6, e_{17}] = e_6, && [e_6, e_{19}] = e_7, && [e_6, e_{22}] = e_{10},
\end{aligned}
$$

$$
\begin{array}{llll}
[e_7, e_{11}] = -e_3, & [e_7, e_{12}] = e_7, & [e_7, e_{17}] = -e_7, & [e_7, e_{20}] = -e_7, \\
[e_7, e_{23}] = -2e_6, & [e_7, e_{24}] = -2e_{10}, & [e_8, e_{11}] = -e_8, & [e_8, e_{13}] = e_8, \\
[e_8, e_{14}] = e_8, & [e_8, e_{16}] = e_9, & [e_9, e_{11}] = -e_9, & [e_9, e_{13}] = e_9, \\
[e_9, e_{14}] = -e_9, & [e_9, e_{15}] = e_8, & [e_{10}, e_{11}] = -e_1 - \frac{e_7}{2}, & [e_{10}, e_{12}] = e_{10}, \\
[e_{10}, e_{18}] = e_7, & [e_{10}, e_{20}] = e_{10}, & [e_{10}, e_{21}] = e_6, & [e_{14}, e_{15}] = -2e_{15}, \\
[e_{14}, e_{16}] = 2e_{16}, & [e_{15}, e_{16}] = -e_{14}, & [e_{17}, e_{18}] = e_{18}, & [e_{17}, e_{19}] = 2e_{19}, \\
[e_{17}, e_{21}] = -e_{21}, & [e_{17}, e_{22}] = e_{22}, & [e_{17}, e_{23}] = -2e_{23}, & [e_{17}, e_{24}] = -e_{24}, \\
[e_{18}, e_{20}] = -2e_{18}, & [e_{18}, e_{22}] = -e_{19}, & [e_{18}, e_{23}] = -2e_{21}, & [e_{18}, e_{24}] = -2e_{20}, \\
[e_{19}, e_{20}] = -e_{19}, & [e_{19}, e_{21}] = -e_{18}, & [e_{19}, e_{23}] = -2e_{17}, & [e_{19}, e_{24}] = -2e_{22}, \\
[e_{20}, e_{21}] = e_{21}, & [e_{20}, e_{22}] = -e_{22}, & [e_{20}, e_{23}] = -e_{23}, & [e_{20}, e_{24}] = -2e_{24}, \\
[e_{21}, e_{22}] = -e_{17} + e_{20}, & [e_{21}, e_{24}] = -e_{23}, & [e_{22}, e_{23}] = -e_{24}. &
\end{array}
$$

The algebra is a $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes \mathbb{R}^{10})$ Levi decomposition algebra. The semisimple factor is a direct sum of $\mathfrak{sl}(2,\mathbb{R})$ spanned by e_{14}, e_{15}, e_{16} and $\mathfrak{sl}(3,\mathbb{R})$ spanned by $e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}, e_{23}, e_{24}$. The radical comprises a ten-dimensional indecomposable nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and a three-dimensional abelian complement spanned by e_{11}, e_{12}, e_{13}.

4.3 Algebra$_{5,9}^{bc}$

The nonzero brackets are

$$[e_1, e_5] = e_1,\ [e_2, e_5] = e_1 + e_2,\ [e_3, e_5] = be_3,\ [e_4, e_5] = ce_4; \quad (bc \neq 0), \tag{4.5}$$

and the associated system of geodesic equations is

$$\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w}, \quad \ddot{z} = c\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.6}$$

The symmetry basis and nonzero brackets are, respectively,

$$
\begin{aligned}
&e_1 = D_x, && e_2 = D_q, \quad e_3 = xD_q, \quad e_4 = e^w D_q, \quad e_5 = (w-1)e^w D_q + e^w D_x,\\
&e_6 = wD_t, && e_7 = D_y, \quad e_8 = D_z, \quad e_9 = e^{bw} D_y, \quad e_{10} = e^{cw} D_z,\\
&e_{11} = D_t, && e_{12} = tD_t, \quad e_{13} = D_w, \quad e_{14} = yD_y, \quad e_{15} = zD_z,\\
&e_{16} = qD_q + xD_x.
\end{aligned}
$$

$$
\begin{aligned}
&[e_1, e_3] = e_2, && [e_1, e_{16}] = e_1, && [e_2, e_{16}] = e_2, && [e_3, e_5] = -e_4, && [e_4, e_{13}] = -e_4,\\
&[e_4, e_{16}] = e_4, && [e_5, e_{13}] = -e_4 - e_5, && [e_5, e_{16}] = e_5, && [e_6, e_{12}] = e_6, && [e_6, e_{13}] = -e_{11},\\
&[e_7, e_{14}] = e_7, && [e_8, e_{15}] = e_8, && [e_9, e_{13}] = -be_9, && [e_9, e_{14}] = e_9, && [e_{10}, e_{13}] = -ce_{10},\\
&[e_{10}, e_{15}] = e_{10}, && [e_{11}, e_{12}] = e_{11}.
\end{aligned}
$$

For the generic case, the symmetry algebra is a sixteen-dimensional indecomposable solvable $\mathbb{R}^5 \rtimes (H_5 \oplus \mathbb{R}^6)$. The non-abelian nilradical is $H_5 \oplus \mathbb{R}^6$. Here H denotes the five-dimensional Heisenberg algebra and is spanned by e_1, e_2, e_3, e_4, e_5, and the $\mathbb{R}^6$ summand is spanned by $e_6, e_7, e_8, e_9, e_{10}, e_{11}$. The complement to the nilradical is abelian spanned by $e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$.

4.3.1 Subcase $b = 1$

The symmetries and nonzero brackets are, respectively,

$$
\begin{aligned}
&e_1 = xD_q, && e_2 = D_x, && e_3 = D_q, && e_4 = xD_z,\\
&e_5 = (w-1)e^w D_q + e^w D_x, && e_6 = e^w D_q, && e_7 = zD_q, && e_8 = e^w D_z,\\
&e_9 = D_z, && e_{10} = D_y, && e_{11} = D_t, && e_{12} = e^{cw} D_y,\\
&e_{13} = wD_t, && e_{14} = tD_t, && e_{15} = D_w, && e_{16} = yD_y,\\
&e_{17} = zD_z, && e_{18} = qD_q + xD_x.
\end{aligned}
$$

$$
\begin{array}{lllll}
[e_1, e_2] = -e_3, & [e_1, e_5] = -e_6, & [e_2, e_{14}] = e_9, & [e_2, e_{18}] = e_2, & [e_3, e_{18}] = e_3, \\
[e_4, e_5] = -e_8, & [e_4, e_7] = e_1, & [e_4, e_{17}] = e_4, & [e_4, e_{18}] = -e_4, & [e_5, e_{15}] = -e_5 - e_6, \\
[e_5, e_{18}] = e_5, & [e_6, e_{15}] = -e_6, & [e_6, e_{18}] = e_6, & [e_7, e_8] = -e_6, & [e_7, e_9] = -e_3, \\
[e_7, e_{17}] = -e_7, & [e_7, e_{18}] = e_7, & [e_8, e_{15}] = -e_8, & [e_8, e_{17}] = e_8, & [e_9, e_{17}] = e_9, \\
[e_{10}, e_{16}] = e_{10}, & [e_{11}, e_{14}] = e_{11}, & [e_{12}, e_{15}] = -ce_{12}, & [e_{12}, e_{16}] = e_{12}, & [e_{13}, e_{14}] = e_{13}, \\
[e_{13}, e_{15}] = -e_{11}. & & & &
\end{array}
$$

4.3.2 Subcase $c = 1$

The symmetries and nonzero brackets are, respectively,

$$
\begin{array}{lllll}
e_1 = xD_q, & e_2 = D_x, & e_3 = D_q, & e_4 = xD_z, & e_5 = (w-1)e^w D_q + e^w D_x, \\
e_6 = e^w D_q, & e_7 = zD_q, & e_8 = e^w D_z, & e_9 = D_z, & e_{10} = D_y, \\
e_{11} = D_t, & e_{12} = e^{bw} D_y, & e_{13} = wD_t, & e_{14} = tD_t, & e_{15} = D_w, \\
e_{16} = yD_y, & e_{17} = zD_z, & e_{18} = qD_q + xD_x. & &
\end{array}
$$

$$
\begin{array}{lllll}
[e_1, e_2] = -e_3, & [e_1, e_5] = -e_6, & [e_2, e_{14}] = e_9, & [e_2, e_{18}] = e_2, & [e_3, e_{18}] = e_3, \\
[e_4, e_5] = -e_8, & [e_4, e_7] = e_1, & [e_4, e_{17}] = e_4, & [e_4, e_{18}] = -e_4, & [e_5, e_{15}] = -e_5 - e_6, \\
[e_5, e_{18}] = e_5, & [e_6, e_{15}] = -e_6, & [e_6, e_{18}] = e_6, & [e_7, e_8] = -e_6, & [e_7, e_9] = -e_3, \\
[e_7, e_{17}] = -e_7, & [e_7, e_{18}] = e_7, & [e_8, e_{15}] = -e_8, & [e_8, e_{17}] = e_8, & [e_9, e_{17}] = e_9, \\
[e_{10}, e_{16}] = e_{10}, & [e_{11}, e_{14}] = e_{11}, & [e_{12}, e_{15}] = -be_{12}, & [e_{12}, e_{16}] = e_{12}, & [e_{13}, e_{14}] = e_{13}, \\
[e_{13}, e_{15}] = -e_{11}. & & & &
\end{array}
$$

For both subcases, it is an eighteen-dimensional indecomposable solvable algebra. The nilrdaical is an non-abelian Lie algebra, $N_9 \oplus \mathbb{R}^4$, where N_9 is a nine-dimensional indecomposable nilpotent spanned by $e_1, e_2, e_3, e_4, e_5 e_6, e_7, e_8, e_9$ and $\mathbb{R}^4$ $e_{10}, e_{11}, e_{12}, e_{13}$. The complement to the nilradical is a five-dimensional abelian spanned by $e_{14}, e_{15}, e_{16}, e_{17}, e_{18}$.

4.3.3 Subcase $b = c$

The symmetries and nonzero brackets are, respectively,

$$
\begin{array}{llll}
e_1 = D_x, & e_2 = e^w D_q, & e_3 = xD_q, & e_4 = (w-1)e^w D_q + e^w D_x, \\
e_5 = D_q, & e_6 = D_z, & e_7 = D_y, & e_8 = D_t, \\
e_9 = e^{cw} D_z, & e_{10} = e^{cw} D_y, & e_{11} = wD_t, & e_{12} = qD_q + xD_x, \\
e_{13} = tD_t, & e_{14} = D_w, & e_{15} = yD_y + zD_z, & e_{16} = yD_y - zD_z, \\
e_{17} = zD_y, & e_{18} = yD_z. & &
\end{array}
$$

$$
\begin{array}{lllll}
[e_1, e_3] = e_5, & [e_1, e_{12}] = e_1, & [e_2, e_{12}] = e_2, & [e_2, e_{14}] = -e_2, & [e_3, e_4] = -e_2, \\
[e_4, e_{12}] = e_4, & [e_4, e_{14}] = -e_2 - e_4, & [e_5, e_{12}] = e_5, & [e_6, e_{15}] = e_6, & [e_6, e_{16}] = -e_6, \\
[e_6, e_{17}] = e_7, & [e_7, e_{15}] = e_7, & [e_7, e_{16}] = e_7, & [e_7, e_{18}] = e_6, & [e_8, e_{13}] = e_8, \\
[e_9, e_{14}] = -ce_9, & [e_9, e_{15}] = e_9, & [e_9, e_{16}] = -e_9, & [e_9, e_{17}] = e_{10}, & [e_{10}, e_{14}] = -ce_{10}, \\
[e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = e_{10}, & [e_{10}, e_{18}] = e_9, & [e_{11}, e_{13}] = e_{11}, & [e_{11}, e_{14}] = -e_8, \\
[e_{16}, e_{17}] = -2e_{17}, & [e_{16}, e_{18}] = 2e_{18}, & [e_{17}, e_{18}] = -e_{16}. & &
\end{array}
$$

The symmetry algebra is $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^4 \ltimes \mathbb{R}^6 \oplus H_5)$ Levi decomposition where the radical consisting of a decomposable nilradical $\mathbb{R}^6 \oplus H_5$ spanned by $e_6, e_7, e_8, e_9, e_{10}, e_{11}$ and e_1, e_2, e_3, e_4, e_5, respectively, as well as an abelain complement spanned by $e_{12}, e_{13}, e_{14}, e_{15}$. The semisimple part is $\mathfrak{sl}(2,\mathbb{R})$ spanned by e_{16}, e_{17}, e_{18}.

4.3.4 Subcase $b = 1, c = 1$

The symmetries and nonzero brackets are, respectively,

$$
\begin{aligned}
&e_1 = D_x, && e_2 = xD_q, && e_3 = xD_y, && e_4 = e^w D_q,\\
&e_5 = (w-1)e^w D_q + e^w D_x, && e_6 = D_t, && e_7 = D_y, && e_8 = e^w D_y,\\
&e_9 = wD_t, && e_{10} = zD_q, && e_{11} = D_q, && e_{12} = yD_q,\\
&e_{13} = D_z, && e_{14} = e^w D_z, && e_{15} = xD_z, && e_{16} = D_w,\\
&e_{17} = tD_t, && e_{18} = qD_q + xD_x, && e_{19} = yD_y + zD_z, && e_{20} = yD_y - zD_z,\\
&e_{21} = zD_y, && e_{22} = yD_z.
\end{aligned}
$$

$$
\begin{aligned}
&[e_1, e_2] = e_{11}, && [e_1, e_3] = e_7, && [e_1, e_{15}] = e_{13}, && [e_1, e_{18}] = e_1, && [e_2, e_5] = -e_4,\\
&[e_3, e_5] = -e_8, && [e_3, e_{12}] = e_2, && [e_3, e_{18}] = -e_3, && [e_3, e_{19}] = e_3, && [e_3, e_{20}] = e_3,\\
&[e_3, e_{22}] = e_{15}, && [e_4, e_{16}] = -e_4, && [e_4, e_{18}] = e_4, && [e_5, e_{15}] = e_{14}, && [e_5, e_{16}] = -e_4 - e_5,\\
&[e_5, e_{18}] = e_5, && [e_6, e_{17}] = e_6, && [e_7, e_{12}] = e_{11}, && [e_7, e_{19}] = e_7, && [e_7, e_{20}] = e_7,\\
&[e_7, e_{22}] = e_{13}, && [e_8, e_{12}] = e_4, && [e_8, e_{16}] = -e_8, && [e_8, e_{19}] = e_8, && [e_8, e_{20}] = e_8,\\
&[e_8, e_{22}] = e_{14}, && [e_9, e_{16}] = -e_6, && [e_9, e_{17}] = e_9, && [e_{10}, e_{13}] = -e_{11}, && [e_{10}, e_{14}] = -e_4,\\
&[e_{10}, e_{15}] = -e_2, && [e_{10}, e_{18}] = e_{10}, && [e_{10}, e_{19}] = -e_{10}, && [e_{10}, e_{20}] = e_{10}, && [e_{10}, e_{22}] = -e_{12},\\
&[e_{11}, e_{18}] = e_{11}, && [e_{12}, e_{18}] = e_{12}, && [e_{12}, e_{19}] = -e_{12}, && [e_{12}, e_{20}] = -e_{12}, && [e_{12}, e_{21}] = -e_{10},\\
&[e_{13}, e_{19}] = e_{13}, && [e_{13}, e_{20}] = -e_{13}, && [e_{13}, e_{21}] = e_7, && [e_{14}, e_{16}] = -e_{14}, && [e_{14}, e_{19}] = e_{14},\\
&[e_{14}, e_{20}] = -e_{14}, && [e_{14}, e_{21}] = e_8, && [e_{15}, e_{18}] = -e_{15}, && [e_{15}, e_{19}] = e_{15}, && [e_{15}, e_{20}] = -e_{15},\\
&[e_{15}, e_{21}] = e_3, && [e_{20}, e_{21}] = -2e_{21}, && [e_{20}, e_{22}] = 2e_{22}, && [e_{21}, e_{22}] = -e_{20}.
\end{aligned}
$$

The symmetry algebra is $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^4 \rtimes \mathbb{R}^{15})$ indecomposable Levi decomposition with a twenty-two-dimensional. It has a nineteen-dimensional solvable consisting of a fifteen-dimensional non-abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$, $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$ and a four-dimensional abelian complement spanned by e_{16}, e_{17}, e_{18}, e_{19}. The $\mathfrak{sl}(2,\mathbb{R})$ part is semisimple spanned by e_{20}, e_{21}, e_{22}.

4.4 Algebra$_{5,10}$

The nonzero brackets are

$$[e_2, e_5] = e_1,\ [e_3, e_5] = e_2,\ [e_4, e_5] = e_3, \tag{4.7}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{x}\dot{w}, \quad \ddot{x} = \dot{y}\dot{w}, \quad \ddot{y} = 0, \quad \ddot{z} = \dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.8}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$e_1 = D_t, \quad e_2 = tD_q, \quad e_3 = D_z, \quad e_4 = D_q, \quad e_5 = D_x, \quad e_6 = D_y, \quad e_7 = wD_t,$$
$$e_8 = yD_t, \quad e_9 = yD_q, \quad e_{10} = wD_q, \quad e_{11} = xD_q + yD_x, \quad e_{12} = e^w D_z,$$
$$e_{13} = \frac{1}{2}w^2 D_q + wD_x, \quad e_{14} = wyD_q + 2yD_x, \quad e_{15} = \frac{1}{6}w^3 D_q + \frac{1}{2}w^2 D_x + wD_y,$$
$$e_{16} = D_w, \quad e_{17} = zD_z, \quad e_{18} = qD_q + xD_x + yD_y - \frac{1}{2}(wx - w^2 y)D_q - \frac{1}{2}(-wy + 2x)D_x,$$
$$e_{19} = tD_t + \frac{1}{2}(wx - w^2 y)D_q + \frac{1}{2}(-wy + 2x)D_x,$$
$$e_{20} = tD_t - \frac{1}{2}(wx - w^2 y)D_q - \frac{1}{2}(-wy + 2x)D_x, \quad e_{21} = twD_q + 2tD_x, \quad e_{22} = (wy - 2x)D_t.$$

$$[e_1, e_2] = e_4, \qquad [e_1, e_{19}] = e_1, \qquad [e_1, e_{20}] = e_1,$$
$$[e_1, e_{21}] = e_{10} + 2e_5, \qquad [e_2, e_7] = -e_{10}, \qquad [e_2, e_8] = -e_9,$$
$$[e_2, e_{18}] = e_2, \qquad [e_2, e_{19}] = -e_2, \qquad [e_2, e_{20}] = -e_2,$$
$$[e_2, e_{22}] = 2e_{11} - e_{14}, \qquad [e_3, e_{17}] = e_3, \qquad [e_4, e_{18}] = e_4,$$
$$[e_5, e_{11}] = e_4, \qquad [e_5, e_{18}] = -\frac{1}{2}e_{10}, \qquad [e_5, e_{19}] = e_5 + \frac{1}{2}e_{10},$$

$$
\begin{array}{lll}
[e_5, e_{20}] = -\frac{1}{2}e_{10} - e_5, & [e_5, e_{22}] = -2e_1, & [e_6, e_8] = e_1, \\
[e_6, e_9] = e_4, & [e_6, e_{11}] = e_5, & [e_6, e_{14}] = e_{10} + 2e_5, \\
[e_6, e_{18}] = e_6 + \frac{1}{2}e_{13}, & [e_6, e_{19}] = -\frac{1}{2}e_{13}, & [e_6, e_{20}] = \frac{1}{2}e_{13}, \\
[e_6, e_{22}] = e_7, & [e_7, e_{16}] = -e_1, & [e_7, e_{19}] = e_7, \\
[e_7, e_{20}] = e_7, & [e_7, e_{21}] = 2e_{13}, & [e_8, e_{15}] = -e_7, \\
[e_8, e_{18}] = -e_8, & [e_8, e_{19}] = e_8, & [e_8, e_{20}] = e_8, \\
[e_8, e_{21}] = e_{14}, & [e_9, e_{15}] = -e_{10}, & [e_{10}, e_{16}] = -e_4, \\
[e_{10}, e_{18}] = e_{10}, & [e_{11}, e_{13}] = -e_{10}, & [e_{11}, e_{14}] = -2e_9, \\
[e_{11}, e_{15}] = -e_{13}, & [e_{11}, e_{18}] = e_{11} - e_{14}, & [e_{11}, e_{19}] = -e_{11} + e_{14}, \\
[e_{11}, e_{20}] = e_{11} - e_{14}, & [e_{11}, e_{21}] = -2e_2, & [e_{11}, e_{22}] = -2e_8, \\
[e_{12}, e_{16}] = -e_{12}, & [e_{12}, e_{17}] = e_{12}, & [e_{13}, e_{16}] = -e_{10} - e_5, \\
[e_{13}, e_{19}] = e_{13}, & [e_{13}, e_{20}] = -e_{13}, & [e_{13}, e_{22}] = -2e_7, \\
[e_{14}, e_{15}] = -2e_{13}, & [e_{14}, e_{16}] = -e_9, & [e_{14}, e_{18}] = -e_{14}, \\
[e_{14}, e_{19}] = e_{14}, & [e_{14}, e_{20}] = -e_{14}, & [e_{14}, e_{22}] = -4e_8, \\
[e_{15}, e_{16}] = -e_{13} - e_6, & [e_{15}, e_{18}] = e_{15}, & [e_{16}, e_{18}] = \frac{1}{2}(-e_{11} + e_{14}), \\
[e_{16}, e_{19}] = \frac{1}{2}(e_{11} - e_{14}), & [e_{16}, e_{20}] = \frac{1}{2}(-e_{11} + e_{14}), & [e_{16}, e_{21}] = e_2, \\
[e_{16}, e_{22}] = e_8, & [e_{20}, e_{21}] = 2e_{21}, & [e_{20}, e_{22}] = -2e_{22}, \\
[e_{21}, e_{22}] = -4e_{20}. & &
\end{array}
$$

The symmetry algebra is a twenty-two-dimensional indecomposable Levi decomposition, where the semisimple is $\mathfrak{sl}(2, \mathbb{R})$ spanned by e_{20}, e_{21}, e_{22} and the nilradical is non-abelian spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$.

4.5 Algebra$_{5,11}^{c}$

The nonzero brackets are

$$[e_1, e_5] = e_1,\ [e_2, e_5] = e_1 + e_2,\ [e_3, e_5] = e_2 + e_3,\ [e_4, e_5] = ce_4; \quad (c \neq 0), \qquad (4.9)$$

and the associated system of geodesic equations is

$$\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = \dot{x}\dot{w} + \dot{y}\dot{w}, \quad \ddot{y} = \dot{y}\dot{w}, \quad \ddot{z} = c\dot{z}\dot{w}, \quad \ddot{w} = 0. \qquad (4.10)$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$e_1 = D_q, \quad e_2 = D_x, \quad e_3 = xD_q + yD_x, \quad e_4 = D_y, \quad e_5 = (w-1)e^w D_q + e^w D_x,$$
$$e_6 = e^w D_q, \quad e_7 = yD_q, \quad e_8 = (\frac{w^2}{2} - w + 1)e^w D_q + (w-1)e^w D_x + e^w D_y,$$
$$e_9 = D_z, \quad e_{10} = e^{cw} D_z, \quad e_{11} = D_t, \quad e_{12} = wD_t, \quad e_{13} = D_w, \quad e_{14} = tD_t,$$
$$e_{15} = zD_z, \quad e_{16} = qD_q + xD_x + yD_y.$$

$$[e_1, e_{16}] = e_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_{16}] = e_2, \quad [e_3, e_4] = -e_2, \quad [e_3, e_5] = -e_6,$$
$$[e_3, e_8] = -e_5, \quad [e_4, e_7] = e_1, \quad [e_4, e_{16}] = e_4, \quad [e_5, e_{13}] = -e_5 - e_6, \quad [e_5, e_{16}] = e_5,$$
$$[e_6, e_{13}] = -e_6, \quad [e_6, e_{16}] = e_6, \quad [e_7, e_8] = -e_6, \quad [e_8, e_{13}] = -e_5 - e_8, \quad [e_8, e_{16}] = e_8,$$
$$[e_9, e_{15}] = e_9, \quad [e_{10}, e_{13}] = -ce_{10}, \quad [e_{10}, e_{15}] = e_{10}, \quad [e_{11}, e_{14}] = e_{11}, \quad [e_{12}, e_{13}] = -e_{11},$$
$$[e_{12}, e_{14}] = e_{12}.$$

For the generic case, the symmetry algebra is a sixteen-dimensional indecomposable solvable Lie algebra $\mathbb{R}^4 \rtimes (N_9 \oplus \mathbb{R}^3)$. The nilrical is composed of a twelve-dimensional decomposable, a direct sum of a nine-dimensional nilpotent spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ and $\mathbb{R}^3$ spanned by e_{10}, e_{11}, e_{12}. The complement to the nilradical is a four-dimensional abelian spanned by $e_{13}, e_{14}, e_{15}, e_{16}$.

4.5.1 Subcase $c = 1$

The symmetries and nonzero brackets are, respectively,

$$e_1 = D_t, \quad e_2 = D_q, \quad e_3 = D_x, \quad e_4 = D_z, \quad e_5 = D_y, \quad e_6 = yD_z, \quad e_7 = yD_q,$$
$$e_8 = zD_q, \quad e_9 = wD_t, \quad e_{10} = xD_q + yD_x, \quad e_{11} = e^w D_q, \quad e_{12} = e^w D_z,$$
$$e_{13} = (w-1)e^w D_q + e^w D_x, \quad e_{14} = (\frac{w^2}{2} - w + 1)e^w D_q + (w-1)e^w D_x + e^w D_y,$$
$$e_{15} = D_w, \quad e_{16} = tD_t, \quad e_{17} = zD_z, \quad e_{18} = qD_q + xD_x + yD_y.$$

$$\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_2, e_{18}] = e_2, & [e_3, e_{10}] = e_2, & [e_3, e_{18}] = e_3, \\
[e_4, e_8] = e_2, & [e_4, e_{17}] = e_4, & [e_5, e_6] = e_4, & [e_5, e_7] = e_2, \\
[e_5, e_{10}] = e_3, & [e_5, e_{18}] = e_5, & [e_6, e_8] = e_7, & [e_6, e_{14}] = -e_{12}, \\
[e_6, e_{17}] = e_6, & [e_6, e_{18}] = -e_6, & [e_7, e_{14}] = -e_{11}, & [e_8, e_{12}] = -e_{11}, \\
[e_8, e_{17}] = -e_8, & [e_8, e_{18}] = e_8, & [e_9, e_{15}] = -e_1, & [e_9, e_{16}] = e_9, \\
[e_{10}, e_{13}] = -e_{11}, & [e_{10}, e_{14}] = -e_{13}, & [e_{11}, e_{15}] = -e_{11}, & [e_{11}, e_{18}] = e_{11}, \\
[e_{12}, e_{15}] = -e_{12}, & [e_{12}, e_{17}] = e_{12}, & [e_{13}, e_{15}] = -e_{11} - e_{13}, & [e_{13}, e_{18}] = e_{13}, \\
[e_{14}, e_{15}] = -e_{13} - e_{14}, & [e_{14}, e_{18}] = e_{14}. & &
\end{array}$$

The symmetry algebra is $\mathbb{R}^4 \ltimes \mathbb{R}^{14}$ indecomposable solvable. It has a fourteen-dimensional non-abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$ and a four-dimensional abelian complement spanned by $e_{15}, e_{16}, e_{17}, e_{18}$.

4.6 Algebra$_{5,12}$

The nonzero brackets are

$$[e_1, e_5] = e_1, \ [e_2, e_5] = e_1 + e_2, \ [e_3, e_5] = e_2 + e_3, \ [e_4, e_5] = e_3 + e_4, \tag{4.11}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = \dot{x}\dot{w} + \dot{y}\dot{w}, \quad \ddot{y} = \dot{y}\dot{w} + \dot{z}\dot{w}, \quad \ddot{z} = \dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.12}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$e_1 = D_q, \quad e_2 = D_y, \quad e_3 = D_x, \quad e_4 = D_z, \quad e_5 = D_t, \quad e_6 = zD_q, \quad e_7 = wD_t,$$

$$e_8 = yD_q + zD_x, \quad e_9 = e^w D_q, \quad e_{10} = xD_q + yD_x + zD_y, \quad e_{11} = (w-1)e^w D_q + e^w D_x,$$

$$e_{12} = (w^2 - 2w + 2)e^w D_q + (2w-2)e^w D_x + 2e^w D_y,$$

$$e_{13} = (\frac{w^3}{6} - \frac{w^2}{2} + w - 1)e^w D_q + (\frac{w^2}{2} - w + 1)e^w D_x + (w-1)e^w D_y + e^w D_z, \quad e_{14} = tD_t,$$

$$e_{15} = D_w, \quad e_{16} = qD_q + xD_x + yD_y + zD_z.$$

$$\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_2, e_8] = e_1, & [e_2, e_{10}] = e_3, & [e_2, e_{16}] = e_2, \\
[e_3, e_{10}] = e_1, & [e_3, e_{16}] = e_3, & [e_4, e_6] = e_1, & [e_4, e_8] = e_3, \\
[e_4, e_{10}] = e_2, & [e_4, e_{16}] = e_4, & [e_5, e_{14}] = e_5, & [e_6, e_{13}] = -e_9, \\
[e_7, e_{14}] = e_7, & [e_7, e_{15}] = -e_5, & [e_8, e_{12}] = -2e_9, & [e_8, e_{13}] = -e_{11}, \\
[e_9, e_{15}] = -e_9, & [e_9, e_{16}] = e_9, & [e_{10}, e_{11}] = -e_9, & [e_{10}, e_{12}] = -2e_{11}, \\
[e_{10}, e_{13}] = -\frac{e_{12}}{2}, & [e_{11}, e_{15}] = -e_{11} - e_9, & [e_{11}, e_{16}] = e_{11}, & [e_{12}, e_{15}] = -2e_{11} - e_{12}, \\
[e_{12}, e_{16}] = e_{12}, & [e_{13}, e_{15}] = -\frac{e_{12}}{2} - e_{13}, & [e_{13}, e_{16}] = e_{13}. &
\end{array}$$

It is a sixteen-dimensional indecomposable solvable algebra. The nilradical is a thirteen-dimensional non-abelian spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}$, where its complement $\mathbb{R}^3$ is abelian spanned by e_{14}, e_{15}, e_{16}.

4.7 Algebra$_{5,13}^{abc}$

The nonzero brackets are

$$[e_1, e_5] = e_1,\ [e_2, e_5] = ae_2,\ [e_3, e_5] = be_3 - ce_4,\ [e_4, e_5] = ce_3 + be_4;\ (ac \neq 0, |a| \leq 1), \tag{4.13}$$

and the associated system of geodesic equations is

$$\ddot{q} = \dot{q}\dot{w}, \quad \ddot{x} = a\dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w} + c\dot{z}\dot{w}, \quad \ddot{z} = -c\dot{y}\dot{w} + b\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.14}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$e_1 = D_y, \quad e_2 = D_z, \quad e_3 = D_q, \quad e_4 = D_x, \quad e_5 = D_t, \quad e_6 = wD_t, \quad e_7 = e^w D_q,$$
$$e_8 = e^{aw} D_x, \quad e_9 = e^{bw}\big(\sin cwD_y + \cos cwD_z\big), \quad e_{10} = e^{bw}\big(\cos cwD_y - \sin cwD_z\big),$$
$$e_{11} = D_w, \quad e_{12} = tD_t, \quad e_{13} = qD_q, \quad e_{14} = xD_x, \quad e_{15} = yD_y + zD_z, \quad e_{16} = zD_y - yD_z.$$

$$\begin{array}{llll}
[e_1, e_{15}] = e_1, & [e_1, e_{16}] = -e_2, & [e_2, e_{15}] = e_2, & [e_2, e_{16}] = e_1, \\
[e_3, e_{13}] = e_3, & [e_4, e_{14}] = e_4, & [e_5, e_{12}] = e_5, & [e_6, e_{11}] = -e_5, \\
[e_6, e_{12}] = e_6, & [e_7, e_{11}] = -e_7, & [e_7, e_{13}] = e_7, & [e_8, e_{11}] = -ae_8, \\
[e_8, e_{14}] = e_8, & [e_9, e_{11}] = -be_9 - ce_{10}, & [e_9, e_{15}] = e_9, & [e_9, e_{16}] = e_{10}, \\
[e_{10}, e_{11}] = -be_{10} + ce_9, & [e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = -e_9. &
\end{array}$$

For the generic case, it is $\mathbb{R}^6 \rtimes \mathbb{R}^{10}$ indecomposable solvable Lie algebra. The nilradical and its complement are abelian spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$, respectively.

4.7.1 Subcase $a = 1$

The symmetries and nonzero brackets are, respectively,

$$e_1 = D_t, \quad e_2 = D_y, \quad e_3 = D_z, \quad e_4 = D_q, \quad e_5 = D_x, \quad e_6 = wD_t, \quad e_7 = e^w D_q,$$

$$e_8 = e^w D_x, \quad e_9 = e^{bw}(\sin cwD_y + \cos cwD_z), \quad e_{10} = e^{bw}(\cos cwD_y - \sin cwD_z),$$

$$e_{11} = D_w, \quad e_{12} = tD_t, \quad e_{13} = yD_y + zD_z, \quad e_{14} = zD_y - yD_z, \quad e_{15} = qD_q + xD_x,$$

$${}_{16} = qD_q - xD_x, \quad e_{17} = xD_q, \quad e_{18} = qD_x.$$

$$\begin{array}{llll}
[e_1, e_{12}] = e_1, & [e_2, e_{13}] = e_2, & [e_2, e_{14}] = -e_3, & [e_3, e_{13}] = e_3, \\
[e_3, e_{14}] = e_2, & [e_4, e_{15}] = e_4, & [e_4, e_{16}] = e_4, & [e_4, e_{18}] = e_5, \\
[e_5, e_{15}] = e_5, & [e_5, e_{16}] = -e_5, & [e_5, e_{17}] = e_4, & [e_6, e_{11}] = -e_1, \\
[e_6, e_{12}] = e_6, & [e_7, e_{11}] = -e_7, & [e_7, e_{15}] = e_7, & [e_7, e_{16}] = e_7, \\
[e_7, e_{18}] = e_8, & [e_8, e_{11}] = -e_8, & [e_8, e_{15}] = e_8, & [e_8, e_{16}] = -e_8, \\
[e_8, e_{17}] = e_7, & [e_9, e_{11}] = -be_9 - ce_{10}, & [e_9, e_{13}] = e_9, & [e_9, e_{14}] = e_{10}, \\
[e_{10}, e_{11}] = -be_{10} + ce_9, & [e_{10}, e_{13}] = e_{10}, & [e_{10}, e_{14}] = -e_9, & [e_{16}, e_{17}] = -2e_{17}, \\
[e_{16}, e_{18}] = 2e_{18}, & [e_{17}, e_{18}] = -e_{16}. & &
\end{array}$$

4.7.2 Subcase $b = 0$

The symmetries and nonzero brackets are, respectively,

$$e_1 = D_t, \quad e_2 = D_y, \quad e_3 = D_z, \quad e_4 = D_q, \quad e_5 = D_x, \quad e_6 = wD_t, \quad e_7 = e^w D_q,$$

$$e_8 = e^{aw} D_x, \quad e_9 = \sin cwD_y + \cos cwD_z, \quad e_{10} = \cos cwD_y - \sin cwD_z,$$

$$e_{11} = D_w + \frac{c}{2}(zD_y - yD_z), \quad e_{12} = tD_t, \quad e_{13} = qD_q, \quad e_{14} = xD_x, \quad e_{15} = yD_y + zD_z,$$

$$e_{16} = zD_y - yD_z, \quad e_{17} = (z\cos cw + y\sin cw)D_y + (y\cos cw - z\sin cw)D_z,$$

$$e_{18} = (y\cos cw - z\sin cw)D_y - (z\cos cw + y\sin cw)D_z.$$

$$
\begin{array}{llll}
[e_1, e_{12}] = e_1, & [e_2, e_{11}] = \frac{-c}{2} e_3, & [e_2, e_{15}] = e_2, & [e_2, e_{16}] = -e_3, \\
[e_2, e_{17}] = e_9, & [e_2, e_{18}] = e_{10}, & [e_3, e_{11}] = \frac{c}{2} e_2, & [e_3, e_{15}] = e_3, \\
[e_3, e_{16}] = e_2, & [e_3, e_{17}] = e_{10}, & [e_3, e_{18}] = -e_9, & [e_4, e_{13}] = e_4, \\
[e_5, e_{14}] = e_5, & [e_6, e_{11}] = -e_1, & [e_6, e_{12}] = e_6, & [e_7, e_{11}] = -e_7, \\
[e_7, e_{13}] = e_7, & [e_8, e_{11}] = -a e_8, & [e_8, e_{14}] = e_8, & [e_9, e_{11}] = \frac{-c}{2} e_{10}, \\
[e_9, e_{15}] = e_9, & [e_9, e_{16}] = e_{10}, & [e_9, e_{17}] = e_2, & [e_9, e_{18}] = -e_3, \\
[e_{10}, e_{11}] = \frac{c}{2} e_9, & [e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = -e_9, & [e_{10}, e_{17}] = e_3, \\
[e_{10}, e_{18}] = e_2, & [e_{16}, e_{17}] = -2e_{18}, & [e_{16}, e_{18}] = 2e_{17}, & [e_{17}, e_{18}] = 2e_{16}.
\end{array}
$$

For both cases, the symmetry algebra is $\mathfrak{sl}(2,\mathbb{R}) \ltimes (\mathbb{R}^5 \ltimes \mathbb{R}^{10})$, where $\mathfrak{sl}(2,\mathbb{R})$ is spanned by e_{16}, e_{17}, e_{18}. The $\mathbb{R}^5$ factor is spanned by $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$ and the nilradical $\mathbb{R}^{10}$ is spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$.

4.7.3 Subcase $b = 1$

The symmetries and nonzero brackets are, respectively,

$$
\begin{gathered}
e_1 = D_y, \quad e_2 = D_z, \quad e_3 = D_q, \quad e_4 = D_x, \quad e_5 = D_t, \quad e_6 = wD_t, \quad e_7 = e^w D_q, \\
e_8 = e^{aw} D_x, \quad e_9 = e^w(\sin cwD_y + \cos cwD_z), \quad e_{10} = e^w(\cos cwD_y - \sin cwD_z), \\
e_{11} = D_w, \quad e_{12} = tD_t, \quad e_{13} = qD_q, \quad e_{14} = xD_x, \quad e_{15} = yD_y + zD_z, \quad e_{16} = zD_y - yD_z.
\end{gathered}
$$

$$
\begin{array}{llll}
[e_1, e_{15}] = e_1, & [e_1, e_{16}] = -e_2, & [e_2, e_{15}] = e_2, & [e_2, e_{16}] = e_1, \\
[e_3, e_{13}] = e_3, & [e_4, e_{14}] = e_4, & [e_5, e_{12}] = e_5, & [e_6, e_{11}] = -e_5, \\
[e_6, e_{12}] = e_6, & [e_7, e_{11}] = -e_7, & [e_7, e_{13}] = e_7, & [e_8, e_{11}] = -a e_8, \\
[e_8, e_{14}] = e_8, & [e_9, e_{11}] = -ce_{10} - e_9, & [e_9, e_{15}] = e_9, & [e_9, e_{16}] = e_{10}, \\
[e_{10}, e_{11}] = ce_9 - e_{10}, & [e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = -e_9. &
\end{array}
$$

The symmetry algebra is $\mathbb{R}^6 \rtimes \mathbb{R}^{10}$ indecomposable solvable where the nilradical and its complement are abelian spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$, respectively.

4.8 Algebra$^a_{5,14}$

The nonzero brackets are

$$[e_2, e_5] = e_1,\ [e_3, e_5] = ae_3 - e_4,\ [e_4, e_5] = e_3 + ae_4, \tag{4.15}$$

and the corresponding system of geodesic equations is

$$\ddot{q} = \dot{x}\dot{w}, \quad \ddot{x} = 0, \quad \ddot{y} = a\dot{y}\dot{w} + \dot{z}\dot{w}, \quad \ddot{z} = -\dot{y}\dot{w} + a\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.16}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned}
&e_1 = D_t, \quad e_2 = e^{aw}(\sin wD_z - \cos wD_y), \quad e_3 = D_q, \quad e_4 = D_z, \quad e_5 = D_y,\\
&e_6 = D_x, \quad e_7 = e^{aw}(\sin wD_y + \cos wD_z), \quad e_8 = wD_q, \quad e_9 = wD_t,\\
&e_{10} = \frac{1}{2}w^2 D_q + wD_x, \quad e_{11} = -zD_y + yD_z, \quad e_{12} = yD_y + zD_z,\\
&e_{13} = qD_q + tD_t + xD_x, \quad e_{14} = D_w + \frac{1}{2}xD_q, \quad e_{15} = tD_q,\\
&e_{16} = tD_t + \frac{1}{2}(wx - 2q)D_q, \quad e_{17} = xD_q, \quad e_{18} = xD_t,\\
&e_{19} = qD_q + xD_x + (wx - 2q)D_q, \quad e_{20} = \frac{1}{2}twD_q + tD_x, \quad e_{21} = (wx - 2q)D_t,\\
&e_{22} = (\frac{1}{2}xw^2 - qw)D_q + (wx - 2q)D_x.
\end{aligned}$$

$$\begin{array}{llll}
[e_1, e_{13}] = e_1, & [e_1, e_{15}] = e_3, & [e_1, e_{16}] = e_1, & [e_1, e_{20}] = e_6 + \frac{1}{2}e_8,\\
[e_2, e_{11}] = -e_7, & [e_2, e_{12}] = e_2, & [e_2, e_{14}] = -ae_2 - e_7, & [e_3, e_{13}] = e_3,\\
[e_3, e_{16}] = -e_3, & [e_3, e_{19}] = -e_3, & [e_3, e_{21}] = -2e_1, & [e_3, e_{22}] = -2e_6 - e_8,\\
[e_4, e_{11}] = -e_5, & [e_4, e_{12}] = e_4, & [e_5, e_{11}] = e_4, & [e_5, e_{12}] = e_5,
\end{array}$$

$$
\begin{array}{llll}
[e_6, e_{13}] = e_6, & [e_6, e_{14}] = \frac{1}{2}e_3, & [e_6, e_{16}] = \frac{1}{2}e_8, & [e_6, e_{17}] = e_3, \\
[e_6, e_{18}] = e_1, & [e_6, e_{19}] = e_6 + e_8, & [e_6, e_{21}] = e_9, & [e_6, e_{22}] = e_{10}, \\
[e_7, e_{11}] = e_2, & [e_7, e_{12}] = e_7, & [e_7, e_{14}] = -ae_7 + e_2, & [e_8, e_{13}] = e_8, \\
[e_8, e_{14}] = -e_3, & [e_8, e_{16}] = -e_8, & [e_8, e_{19}] = -e_8, & [e_8, e_{21}] = -2e_9, \\
[e_8, e_{22}] = -2e_{10}, & [e_9, e_{13}] = e_9, & [e_9, e_{14}] = -e_1, & [e_9, e_{15}] = e_8, \\
[e_9, e_{16}] = e_9, & [e_9, e_{20}] = e_{10}, & [e_{10}, e_{13}] = e_{10}, & [e_{10}, e_{14}] = -e_6 - \frac{1}{2}e_8, \\
[e_{10}, e_{17}] = e_8, & [e_{10}, e_{18}] = e_9, & [e_{10}, e_{19}] = e_{10}, & [e_{15}, e_{16}] = -2e_{15}, \\
[e_{15}, e_{18}] = -e_{17}, & [e_{15}, e_{19}] = -e_{15}, & [e_{15}, e_{21}] = -2e_{16}, & [e_{15}, e_{22}] = -2e_{20}, \\
[e_{16}, e_{17}] = e_{17}, & [e_{16}, e_{18}] = -e_{18}, & [e_{16}, e_{20}] = e_{20}, & [e_{16}, e_{21}] = -2e_{21}, \\
[e_{16}, e_{22}] = -e_{22}, & [e_{17}, e_{19}] = -2e_{17}, & [e_{17}, e_{20}] = -e_{15}, & [e_{17}, e_{21}] = -2e_{18}, \\
[e_{17}, e_{22}] = -2e_{19}, & [e_{18}, e_{19}] = -e_{18}, & [e_{18}, e_{20}] = -e_{16} + e_{19}, & [e_{18}, e_{22}] = -e_{21}, \\
[e_{19}, e_{20}] = -e_{20}, & [e_{19}, e_{21}] = -e_{21}, & [e_{19}, e_{22}] = -2e_{22}, & [e_{20}, e_{21}] = -e_{22}.
\end{array}
$$

For the generic case, the symmetry algebra has a $\mathfrak{sl}(3, \mathbb{R}) \ltimes (\mathbb{R}^4 \rtimes \mathbb{R}^{10})$ Levi decomposition, where the semisimple part is $\mathfrak{sl}(3, \mathbb{R})$ spanned by $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$. The radical $\mathbb{R}^4 \rtimes \mathbb{R}^{10}$ is spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and $e_{11}, e_{12}, e_{13}, e_{14}$.

4.8.1 Subcase $a = 0$

The symmetries and nonzero brackets are, respectively,

$$
\begin{aligned}
&e_1 = D_t, \quad e_2 = D_q, \quad e_3 = D_z, \quad e_4 = D_y, \quad e_5 = D_x, \quad e_6 = \sin w D_z - \cos w D_y,\\
&e_7 = \sin w D_y + \cos w D_z, \quad e_8 = w D_q, \quad e_9 = w D_t, \quad e_{10} = \frac{1}{2} w^2 D_q + w D_x,\\
&e_{11} = y D_y + z D_z, \quad e_{12} = q D_q + t D_t + x D_x, \quad e_{13} = D_w + \frac{1}{2}(x D_q + z D_y - y D_z),\\
&e_{14} = t D_q, \quad e_{15} = t D_t + \frac{1}{2}(wx - 2q) D_q, \quad e_{16} = x D_t, \quad e_{17} = x D_q,\\
&e_{18} = (\frac{1}{2} x w^2 - q w) D_q + (wx - 2q) D_x, \quad e_{19} = \frac{1}{2} t w D_q + t D_x,\\
&e_{20} = q D_q + x D_x + (wx - 2q) D_q, \quad e_{21} = (wx - 2q) D_t, \quad e_{22} = -z D_y + y D_z,\\
&e_{23} = (z \cos w + y \sin w) D_y + (y \cos w - z \sin w) D_z,\\
&e_{24} = (-y \cos w + z \sin w) D_y + (z \cos w + y \sin w) D_z.
\end{aligned}
$$

$$
\begin{array}{llll}
[e_1, e_{12}] = e_1, & [e_1, e_{14}] = e_2, & [e_1, e_{15}] = e_1, & [e_1, e_{19}] = e_5 + \frac{1}{2} e_8,\\
[e_2, e_{12}] = e_2, & [e_2, e_{15}] = -e_2, & [e_2, e_{18}] = -2e_5 - e_8, & [e_2, e_{20}] = -e_2,\\
[e_2, e_{21}] = -2e_1, & [e_3, e_{11}] = e_3, & [e_3, e_{13}] = \frac{1}{2} e_4, & [e_3, e_{22}] = -e_4,\\
[e_3, e_{23}] = -e_6, & [e_3, e_{24}] = e_7, & [e_4, e_{11}] = e_4, & [e_4, e_{13}] = -\frac{1}{2} e_3,\\
[e_4, e_{22}] = e_3, & [e_4, e_{23}] = e_7, & [e_4, e_{24}] = e_6, & [e_5, e_{12}] = e_5,\\
[e_5, e_{13}] = \frac{1}{2} e_2, & [e_5, e_{15}] = \frac{1}{2} e_8, & [e_5, e_{16}] = e_1, & [e_5, e_{17}] = e_2,\\
[e_5, e_{18}] = e_{10}, & [e_5, e_{20}] = e_5 + e_8, & [e_5, e_{21}] = e_9, & [e_6, e_{11}] = e_6,\\
[e_6, e_{13}] = -\frac{1}{2} e_7, & [e_6, e_{22}] = -e_7, & [e_6, e_{23}] = -e_3, & [e_6, e_{24}] = e_4,\\
[e_7, e_{11}] = e_7, & [e_7, e_{13}] = \frac{1}{2} e_6, & [e_7, e_{22}] = e_6, & [e_7, e_{23}] = e_4,\\
[e_7, e_{24}] = e_3, & [e_8, e_{12}] = e_8, & [e_8, e_{13}] = -e_2, & [e_8, e_{15}] = -e_8,\\
[e_8, e_{18}] = -2e_{10}, & [e_8, e_{20}] = -e_8, & [e_8, e_{21}] = -2e_9, & [e_9, e_{12}] = e_9,
\end{array}
$$

$$
\begin{array}{llll}
[e_9, e_{13}] = -e_1, & [e_9, e_{14}] = e_8, & [e_9, e_{15}] = e_9, & [e_9, e_{19}] = e_{10}, \\
[e_{10}, e_{12}] = e_{10}, & [e_{10}, e_{13}] = -e_5 - \frac{1}{2}e_8, & [e_{10}, e_{16}] = e_9, & [e_{10}, e_{17}] = e_8, \\
[e_{10}, e_{20}] = e_{10}, & [e_{14}, e_{15}] = -2e_{14}, & [e_{14}, e_{16}] = -e_{17}, & [e_{14}, e_{18}] = -2e_{19}, \\
[e_{14}, e_{20}] = -e_{14}, & [e_{14}, e_{21}] = -2e_{15}, & [e_{15}, e_{16}] = -e_{16}, & [e_{15}, e_{17}] = e_{17}, \\
[e_{15}, e_{18}] = -e_{18}, & [e_{15}, e_{19}] = e_{19}, & [e_{15}, e_{21}] = -2e_{21}, & [e_{16}, e_{18}] = -e_{21}, \\
[e_{16}, e_{19}] = e_{20} - e_{15}, & [e_{16}, e_{20}] = -e_{16}, & [e_{17}, e_{18}] = -2e_{20}, & [e_{17}, e_{19}] = -e_{14}, \\
[e_{17}, e_{20}] = -2e_{17}, & [e_{17}, e_{21}] = -2e_{16}, & [e_{18}, e_{20}] = 2e_{18}, & [e_{19}, e_{20}] = e_{19}, \\
[e_{19}, e_{21}] = -e_{18}, & [e_{20}, e_{21}] = -e_{21}, & [e_{22}, e_{23}] = -2e_{24}, & [e_{22}, e_{24}] = 2e_{23}, \\
[e_{23}, e_{24}] = 2e_{22}. & & &
\end{array}
$$

The symmetry algebra has a $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes \mathbb{R}^{10})$ Levi decomposition algebra. The semisimple part is $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ spanned by $e_{14}, e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}$ and e_{22}, e_{23}, e_{24}, respectively. The solvable comprises of a three-dimensional abelian complement spanned by e_{11}, e_{12}, e_{13} and a ten-dimensional abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_9, e_{10}$.

4.9 Algebra$^a_{5,15}$

The nonzero brackets are

$$
[e_1, e_5] = e_1,\ [e_2, e_5] = e_1 + e_2,\ [e_3, e_5] = ae_3,\ [e_4, e_5] = e_3 + ae_4; \quad (|a| \leq 1), \tag{4.17}
$$

and the associated system of geodesic equations is

$$
\ddot{q} = \dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = \dot{x}\dot{w}, \quad \ddot{y} = a\dot{y}\dot{w} + \dot{z}\dot{y}, \quad \ddot{z} = a\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.18}
$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$e_1 = D_x, \quad e_2 = D_q, \quad e_3 = xD_q, \quad e_4 = (w-1)e^w D_q + e^w D_x, \quad e_5 = e^w D_q,$$
$$e_6 = \frac{e^{aw}}{a} D_y, \quad e_7 = \frac{(aw-1)e^{aw}}{a} D_y + e^{aw} D_z, \quad e_8 = zD_y, \quad e_9 = D_z, \quad e_{10} = D_y,$$
$$e_{11} = D_t, \quad e_{12} = wD_t, \quad e_{13} = D_w, \quad e_{14} = tD_t, \quad e_{15} = qD_q + xD_x, \quad e_{16} = yD_y + zD_z.$$

$$\begin{array}{llll}
[e_1, e_3] = e_2, & [e_1, e_{15}] = e_1, & [e_2, e_{15}] = e_2, & [e_3, e_4] = -e_5, \\
[e_4, e_{13}] = -e_4 - e_5, & [e_4, e_{15}] = e_4, & [e_5, e_{13}] = -e_5, & [e_5, e_{15}] = e_5, \\
[e_6, e_{13}] = -ae_6, & [e_6, e_{16}] = e_6, & [e_7, e_8] = ae_6, & [e_7, e_{13}] = -ae_6 - ae_7, \\
[e_7, e_{16}] = e_7, & [e_8, e_9] = -e_{10}, & [e_9, e_{16}] = e_9, & [e_{10}, e_{16}] = e_{10}, \\
[e_{11}, e_{14}] = e_{11}, & [e_{12}, e_{13}] = -e_{11}, & [e_{12}, e_{14}] = e_{12}. &
\end{array}$$

For the generic case, the symmetry algebra is $\mathbb{R}^4 \rtimes (H_5 \oplus N_7)$ indecomposable solvable, where the decomposable nilradical is comprised of a five-dimensional Heisenberg spanned by e_1, e_2, e_3, e_4, e_5 and a seven-dimensional nilpotent spanned by $e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$. The $\mathbb{R}^4$ factor is abelian spanned by $e_{13}, e_{14}, e_{15}, e_{15}$.

4.9.1 Subcase $a = 0$

The symmetries and nonzero brackets are, respectively,

$$e_1 = e^w D_q, \quad e_2 = D_q, \quad e_3 = D_x, \quad e_4 = xD_q, \quad e_5 = (w-1)e^w D_q + e^w D_x,$$
$$e_6 = wD_t, \quad e_7 = wD_y, \quad e_8 = D_y, \quad e_9 = D_z, \quad e_{10} = \frac{1}{2}w^2 D_y + wD_z,$$
$$e_{11} = D_t, \quad e_{12} = tD_t + \frac{1}{2}wzD_y + zD_z + (y - \frac{wz}{2})D_y, \quad e_{13} = qD_q + xD_x,$$
$$e_{14} = D_w + \frac{1}{2}zD_y, \quad e_{15} = tD_y, \quad e_{16} = tD_t - (y - \frac{1}{2}wz)D_y, \quad e_{17} = zD_t,$$
$$e_{18} = zD_y, \quad e_{19} = \frac{1}{2}twD_y + tD_z, \quad e_{20} = (wz - 2y)D_t,$$
$$e_{21} = \frac{1}{2}wzD_y + zD_z - (y - \frac{wz}{2})D_y, \quad e_{22} = (\frac{1}{2}zw^2 - wy)D_y + (wz - 2y)D_z.$$

$$
\begin{array}{llll}
[e_1,e_{13}]=e_1, & [e_1,e_{14}]=-e_1, & [e_2,e_{13}]=e_2, & [e_3,e_4]=e_2,\\
[e_3,e_{13}]=e_3, & [e_4,e_5]=-e_1, & [e_5,e_{13}]=e_5, & [e_5,e_{14}]=-e_1-e_5,\\
[e_6,e_{12}]=e_6, & [e_6,e_{14}]=-e_{11}, & [e_6,e_{15}]=e_7, & [e_6,e_{16}]=e_6,\\
[e_6,e_{19}]=e_{10}, & [e_7,e_{12}]=e_7, & [e_7,e_{14}]=-e_8, & [e_7,e_{16}]=-e_7,\\
[e_7,e_{20}]=-2e_6, & [e_7,e_{21}]=-e_7, & [e_7,e_{22}]=-2e_{10}, & [e_8,e_{12}]=e_8,\\
[e_8,e_{16}]=-e_8, & [e_8,e_{20}]=-2e_{11}, & [e_8,e_{21}]=-e_8, & [e_8,e_{22}]=-e_7-2e_9,\\
[e_9,e_{12}]=e_9, & [e_9,e_{14}]=\frac{1}{2}e_8, & [e_9,e_{16}]=\frac{1}{2}e_7, & [e_9,e_{17}]=e_{11},\\
[e_9,e_{18}]=e_8, & [e_9,e_{20}]=e_6, & [e_9,e_{21}]=e_7+e_9, & [e_9,e_{22}]=e_{10},\\
[e_{10},e_{12}]=e_{10}, & [e_{10},e_{14}]=-\frac{1}{2}e_7-e_9, & [e_{10},e_{17}]=e_6, & [e_{10},e_{18}]=e_7,\\
[e_{10},e_{21}]=e_{10}, & [e_{11},e_{12}]=e_{11}, & [e_{11},e_{15}]=e_8, & [e_{11},e_{16}]=e_{11},\\
[e_{11},e_{19}]=\frac{1}{2}e_7+e_9, & [e_{15},e_{16}]=-2e_{15}, & [e_{15},e_{17}]=-e_{18}, & [e_{15},e_{20}]=-2e_{16},\\
[e_{15},e_{21}]=-e_{15}, & [e_{15},e_{22}]=-2e_{19}, & [e_{16},e_{17}]=-e_{17}, & [e_{16},e_{18}]=e_{18},\\
[e_{16},e_{19}]=e_{19}, & [e_{16},e_{20}]=-2e_{20}, & [e_{16},e_{22}]=-e_{22}, & [e_{17},e_{19}]=-e_{16}+e_{21},\\
[e_{17},e_{21}]=-e_{17}, & [e_{17},e_{22}]=-e_{20}, & [e_{18},e_{19}]=-e_{15}, & [e_{18},e_{20}]=-2e_{17},\\
[e_{18},e_{21}]=-2e_{18}, & [e_{18},e_{22}]=-2e_{21}, & [e_{19},e_{20}]=-e_{22}, & [e_{19},e_{21}]=e_{19},\\
[e_{20},e_{21}]=e_{20}, & [e_{21},e_{22}]=-2e_{22}. & &
\end{array}
$$

The symmetry algebra is a twenty-two-dimensional indecomposable, $\mathfrak{sl}(3,\mathbb{R}) \rtimes (\mathbb{R}^3 \rtimes H_5 \oplus \mathbb{R}^6)$. The radical is a semi-direct product $\mathbb{R}^3 \rtimes (H_5 \oplus \mathbb{R}^6)$ with an eleven-dimensional non-abelian nilradical spanned by e_1, e_2, e_3, e_4, e_5, and $e_6, e_7, e_8, e_9, e_{10}, e_{11}$, respectively, as well as a three-dimensional abelian complement spanned by e_{12}, e_{13}, e_{14}. The semisimple part is $\mathfrak{sl}(3,\mathbb{R})$ spanned by $e_{15}, e_{16}, e_{17}, e_{18}, e_{19}, e_{20}, e_{21}, e_{22}$.

4.9.2 Subcase $a = 1$

The symmetries and nonzero brackets are, respectively,

$$e_1 = D_t, \quad e_2 = D_q, \quad e_3 = D_y, \quad e_4 = D_x, \quad e_5 = D_z, \quad e_6 = wD_t,$$

$$e_7 = xD_q, \quad e_8 = zD_q, e_9 = xD_y, \quad e_{10} = zD_y, \quad e_{11} = e^w D_q, \quad e_{12} = e^w D_y,$$

$$e_{13} = (w-1)e^w D_q + e^w D_x, \quad e_{14} = (w-1)e^w D_y + e^w D_z, \quad e_{15} = D_w, \quad e_{16} = tD_t,$$

$$e_{17} = qD_q + xD_x + yD_y + zD_z, \quad e_{18} = qD_q + xD_x - yD_y - zD_z, \quad e_{19} = yD_q + zD_x,$$

$$e_{20} = qD_y + xD_z.$$

$$\begin{array}{llll}
[e_1, e_{16}] = e_1, & [e_2, e_{17}] = e_2, & [e_2, e_{18}] = e_2, & [e_2, e_{20}] = e_3, \\
[e_3, e_{17}] = e_3, & [e_3, e_{18}] = -e_3, & [e_3, e_{19}] = e_2, & [e_4, e_7] = e_2, \\
[e_4, e_9] = e_3, & [e_4, e_{17}] = e_4, & [e_4, e_{18}] = e_4, & [e_4, e_{20}] = e_5, \\
[e_5, e_8] = e_2, & [e_5, e_{10}] = e_3, & [e_5, e_{17}] = e_5, & [e_5, e_{18}] = -e_5, \\
[e_5, e_{19}] = e_4, & [e_6, e_{15}] = -e_1, & [e_6, e_{16}] = e_6, & [e_7, e_{13}] = -e_{11}, \\
[e_7, e_{19}] = -e_8, & [e_7, e_{20}] = e_9, & [e_8, e_{14}] = -e_{11}, & [e_8, e_{18}] = 2e_8, \\
[e_8, e_{20}] = e_{10} - e_7, & [e_9, e_{13}] = -e_{12}, & [e_9, e_{18}] = -2e_9, & [e_9, e_{19}] = -e_{10} + e_7, \\
[e_{10}, e_{14}] = -e_{12}, & [e_{10}, e_{19}] = e_8, & [e_{10}, e_{20}] = -e_9, & [e_{11}, e_{15}] = -e_{11}, \\
[e_{11}, e_{17}] = e_{11}, & [e_{11}, e_{18}] = e_{11}, & [e_{11}, e_{20}] = e_{12}, & [e_{12}, e_{15}] = -e_{12}, \\
[e_{12}, e_{17}] = e_{12}, & [e_{12}, e_{18}] = -e_{12}, & [e_{12}, e_{19}] = e_{11}, & [e_{13}, e_{15}] = -e_{11} - e_{13}, \\
[e_{13}, e_{17}] = e_{13}, & [e_{13}, e_{18}] = e_{13}, & [e_{13}, e_{20}] = e_{14}, & [e_{14}, e_{15}] = -e_{12} - e_{14}, \\
[e_{14}, e_{17}] = e_{14}, & [e_{14}, e_{18}] = -e_{14}, & [e_{14}, e_{19}] = e_{13}, & [e_{18}, e_{19}] = -2e_{19}, \\
[e_{18}, e_{20}] = 2e_{20}, & [e_{19}, e_{20}] = -e_{18}. & &
\end{array}$$

It is a twenty-dimensional indecomposable. The semismple part is $\mathfrak{sl}(2,\mathbb{R})$ spanned by e_{18}, e_{19}, e_{20}. The radical constitutes a three-dimensional abelain complement and a fourteen-dimensional non-abelian nilradical spanned by e_{15}, e_{16}, e_{17} and $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$, individually.

4.10 Algebra$^{ab}_{5,16}$

The nonzero brackets are

$$[e_1,e_5]=e_1,\ [e_2,e_5]=e_1+e_2,\ [e_3,e_5]=ae_3-be_4,\ [e_4,e_5]=be_3+ae_4; \quad (b\neq 0), \tag{4.19}$$

and the corresponding system of geodesic equations is

$$\ddot{q}=\dot{q}\dot{w}+\dot{x}\dot{w}, \quad \ddot{x}=\dot{x}\dot{w}, \quad \ddot{y}=a\dot{y}\dot{w}+b\dot{z}\dot{w}, \quad \ddot{z}=-b\dot{y}\dot{w}+a\dot{z}\dot{w}, \quad \ddot{w}=0. \tag{4.20}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{array}{ll}
e_1=D_q, & e_2=D_z,\\
e_3=D_y, & e_4=D_x,\\
e_5=e^wD_q, & e_6=xD_q,\\
e_7=(w-1)e^wD_q+e^wD_x, & e_8=e^{aw}(\cos bwD_y-\sin bwD_z),\\
e_9=e^{aw}(\sin bwD_y+\cos bwD_z), & e_{10}=D_t,\\
e_{11}=wD_t, & e_{12}=qD_q+xD_x,\\
e_{13}=zD_y-yD_z, & e_{14}=yD_y+zD_z,\\
e_{15}=tD_t, & e_{16}=D_w.
\end{array}$$

$$\begin{array}{llll}
[e_1,e_{12}]=e_1, & [e_2,e_{13}]=e_3, & [e_2,e_{14}]=e_2, & [e_3,e_{13}]=-e_2,\\
[e_3,e_{14}]=e_3, & [e_4,e_6]=e_1, & [e_4,e_{12}]=e_4, & [e_5,e_{12}]=e_5,
\end{array}$$

$$[e_5, e_{16}] = -e_5, \quad [e_6, e_7] = -e_5, \quad [e_7, e_{12}] = e_7, \quad [e_7, e_{16}] = -e_5 - e_7,$$
$$[e_8, e_{13}] = -e_9, \quad [e_8, e_{14}] = e_8, \quad [e_8, e_{16}] = -ae_8 + be_9, \quad [e_9, e_{13}] = e_8,$$
$$[e_9, e_{14}] = e_9, \quad [e_9, e_{16}] = -ae_9 - be_8, \quad [e_{10}, e_{15}] = e_{10}, \quad [e_{11}, e_{15}] = e_{11},$$
$$[e_{11}, e_{16}] = -e_{10}.$$

For the generic case, the symmetry algebra is a sixteen-dimensional indecomposable solvable. The nilradical is an eleven-dimensional decomposeable, $N_7 \oplus \mathbb{R}^4$, spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and e_8, e_9, e_{10}, e_{11}. The complement to the nilradical is a five-dimensional abelian spanned by$e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$.

4.10.1 Subcase $a = 0$

The symmetries and nonzero brackets are, respectively,

$$e_1 = D_q, \qquad e_2 = (w-1)e^w D_q + e^w D_x,$$
$$e_3 = xD_q, \qquad e_4 = D_x,$$
$$e_5 = e^w D_q, \qquad e_6 = wD_t,$$
$$e_7 = D_z, \qquad e_8 = D_t,$$
$$e_9 = \sin bw D_y + \cos bw D_z, \qquad e_{10} = \cos bw D_y - \sin bw D_z,$$
$$e_{11} = D_y, \qquad e_{12} = tD_t,$$
$$e_{13} = D_w + \frac{b}{2}(zD_y - yD_z), \qquad e_{14} = qD_q + xD_x,$$
$$e_{15} = yD_y + zD_z, \qquad e_{16} = zD_y - yD_z,$$
$$e_{17} = (z\cos bw + y\sin bw)D_y + (y\cos bw - z\sin bw)D_z,$$
$$e_{18} = (y\cos bw - z\sin bw)D_y + (-z\cos bw - y\sin bw)D_z.$$

$$[e_1, e_{14}] = e_1, \quad [e_2, e_3] = e_5, \quad [e_2, e_{13}] = -e_2 - e_5, \quad [e_2, e_{14}] = e_2,$$

$$
\begin{array}{llll}
[e_3, e_4] = -e_1, & [e_4, e_{14}] = e_4, & [e_5, e_{13}] = -e_5, & [e_5, e_{14}] = e_5, \\
[e_6, e_{12}] = e_6, & [e_6, e_{13}] = -e_8, & [e_7, e_{13}] = \dfrac{b}{2} e_{11}, & [e_7, e_{15}] = e_7, \\
[e_7, e_{16}] = e_{11}, & [e_7, e_{17}] = e_{10}, & [e_7, e_{18}] = -e_9, & [e_8, e_{12}] = e_8, \\
[e_9, e_{13}] = \dfrac{-b}{2} e_{10}, & [e_9, e_{15}] = e_9, & [e_9, e_{16}] = e_{10}, & [e_9, e_{17}] = e_{11}, \\
[e_9, e_{18}] = -e_7, & [e_{10}, e_{13}] = \dfrac{b}{2} e_9, & [e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = -e_9, \\
[e_{10}, e_{17}] = e_7, & [e_{10}, e_{18}] = e_{11}, & [e_{11}, e_{13}] = -\dfrac{b}{2} e_7, & [e_{11}, e_{15}] = e_{11}, \\
[e_{11}, e_{16}] = -e_7, & [e_{11}, e_{17}] = e_9, & [e_{11}, e_{18}] = e_{10}, & [e_{16}, e_{17}] = -2e_{18}, \\
[e_{16}, e_{18}] = 2e_{17}, & [e_{17}, e_{18}] = 2e_{16}. & &
\end{array}
$$

They symmetry algebra is an eighteen-dimentional indecomposable Levi decomposition. The semisimple is $\mathfrak{sl}(2, \mathbb{R})$ spanned by e_{16}, e_{17}, e_{18}, whereas the radical comprises of an eleven-dimensional decomposable nilradical $H_5 \oplus \mathbb{R}^6$ spanned by e_1, e_2, e_3, e_4, e_5 and $e_7, e_8, e_9, e_{10}, e_{11}$, respectively, as well as a four-dimensional abelian complement spanned by $e_{12}, e_{13}, e_{14}, e_{15}$.

4.10.2 Subcase $a = 1$

The symmetries and nonzero brackets are, respectively,

$$
\begin{array}{lll}
e_1 = D_q, & e_2 = (w-1)e^w D_q + e^w D_x, & e_3 = x D_q, \\
e_4 = D_x, & e_5 = e^w D_q, & e_6 = w D_t, \\
e_7 = D_z, & e_8 = D_t, & e_9 = D_y, \\
e_{10} = e^w(\sin bw D_y + \cos bw D_z), & e_{11} = e^w(\cos bw D_y - \sin bw D_z), & e_{12} = t D_t, \\
e_{13} = D_w, & e_{14} = q D_q + x D_x, & e_{15} = y D_y + z D_z, \\
e_{16} = z D_y - y D_z. & &
\end{array}
$$

$$
\begin{array}{llll}
[e_1, e_{14}] = e_1, & [e_2, e_3] = e_5, & [e_2, e_{13}] = -e_2 - e_5, & [e_2, e_{14}] = e_2, \\
[e_3, e_4] = -e_1, & [e_4, e_{14}] = e_4, & [e_5, e_{13}] = -e_5, & [e_5, e_{14}] = e_5, \\
[e_6, e_{12}] = e_6, & [e_6, e_{13}] = -e8, & [e_7, e_{15}] = e_7, & [e_7, e_{16}] = e_9, \\
[e_8, e_{12}] = e_8, & [e_9, e_{15}] = e_9, & [e_9, e_{16}] = -e_7, & [e_{10}, e_{13}] = -be_{11} - e_{10}, \\
[e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = e_{11}, & [e_{11}, e_{13}] = be_{10} - e_{11}, & [e_{11}, e_{15}] = e_{11}, \\
[e_{11}, e_{16}] = -e_{10}. & & &
\end{array}
$$

4.10.3 Subcase $a = 1, b = 1$

The symmetries and nonzero brackets are, respectively,

$$
\begin{array}{lll}
e_1 = D_q, & e_2 = (w-1)e^w(D_q + D_x), & e_3 = xD_q, \\
e_4 = D_x, & e_5 = e^w D_q, & e_6 = wD_t, \\
e_7 = D_y, & e_8 = D_t, & e_9 = D_z, \\
e_{10} = e^w(\sin wD_y + \cos wD_z), & e_{11} = e^w(\sin wD_z - \cos wD_y), & e_{12} = tD_t, \\
e_{13} = D_w, & e_{14} = qD_q + xD_x, & e_{15} = yD_y + zD_z, \\
e_{16} = yD_z - zD_y. & &
\end{array}
$$

$$
\begin{array}{llll}
[e_1, e_{14}] = e_1, & [e_2, e_3] = e_5, & [e_2, e_{13}] = -e_2 - e_5, & [e_2, e_{14}] = e_2, \\
[e_3, e_4] = -e_1, & [e_4, e_{14}] = e_4, & [e_5, e_{13}] = -e_5, & [e_5, e_{14}] = e_5, \\
[e_6, e_{12}] = e_6, & [e_6, e_{13}] = -e_8, & [e_7, e_{15}] = e_7, & [e_7, e_{16}] = e_9, \\
[e_8, e_{12}] = e_8, & [e_9, e_{15}] = e_9, & [e_9, e_{16}] = -e_7, & [e_{10}, e_{13}] = -e_{10} + e_{11}, \\
[e_{10}, e_{15}] = e_{10}, & [e_{10}, e_{16}] = e_{11}, & [e_{11}, e_{13}] = -e_{10} - e_{11}, & [e_{11}, e_{15}] = e_{11}, \\
[e_{11}, e_{16}] = -e_{10}. & & &
\end{array}
$$

For both subcases, the symmetry algebra is $\mathbb{R}^5 \rtimes (\mathbb{R}^6 \oplus H_5)$ indecomposable solvable. Its nilradical is an eleven-dimensional non-abelian spanned by e_1, e_2, e_3, e_4, e_5 and $e_6, e_7, e_8, e_9, e_{10}, e_{11}$. The complement to the nilrdaical is abelian spanned by $e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$.

4.11 Algebra$_{5,17}^{abc}$

The nonzero brackets are

$$[e_1, e_5] = ae_1 - e_2,\ [e_2, e_5] = e_1 + ae_2,\ [e_3, e_5] = be_3 - ce_4,\ [e_4, e_5] = ce_3 + be_4;\ (c \neq 0), \tag{4.21}$$

and the associated system of geodesic equations is

$$\ddot{q} = a\dot{q}\dot{w} + \dot{x}\dot{w}, \quad \ddot{x} = -\dot{q}\dot{w} + a\dot{x}\dot{w}, \quad \ddot{y} = b\dot{y}\dot{w} + c\dot{z}\dot{w}, \quad \ddot{z} = -c\dot{y}\dot{w} + b\dot{z}\dot{w}, \quad \ddot{w} = 0. \tag{4.22}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned}
e_1 &= e^{bw}(\cos cwD_y - \sin cwD_z), & e_2 &= D_y,\\
e_3 &= D_z, & e_4 &= D_q,\\
e_5 &= D_t, & e_6 &= e^{bw}(\sin cwD_y + \cos cwD_z),\\
e_7 &= D_x, & e_8 &= wD_t,\\
e_9 &= e^{aw}(-\cos wD_q + \sin wD_x), & e_{10} &= e^{aw}(\sin wD_q + \cos wD_x),\\
e_{11} &= yD_y + zD_z, & e_{12} &= qD_q + xD_x,\\
e_{13} &= -xD_q + qD_x, & e_{14} &= zD_y - yD_z,\\
e_{15} &= D_w, & e_{16} &= tD_t.
\end{aligned}$$

$$[e_1, e_{11}] = e_1, \qquad [e_1, e_{14}] = -e_6, \quad [e_1, e_{15}] = -be_1 + ce_6, \quad [e_2, e_{11}] = e_2,$$

$$
\begin{array}{llll}
[e_2, e_{14}] = -e_3, & [e_3, e_{11}] = e_3, & [e_3, e_{14}] = e_2, & [e_4, e_{12}] = e_4, \\
[e_4, e_{13}] = e_7, & [e_5, e_{16}] = e_5, & [e_6, e_{11}] = e_6, & [e_6, e_{14}] = e_1, \\
[e_6, e_{15}] = -be_6 - ce_1, & [e_7, e_{12}] = e_7, & [e_7, e_{13}] = -e_4, & [e_8, e_{15}] = -e_5, \\
[e_8, e_{16}] = e_8, & [e_9, e_{12}] = e_9, & [e_9, e_{13}] = -e_{10}, & [e_9, e_{15}] = -ae_9 - e_{10}, \\
[e_{10}, e_{12}] = e_{10}, & [e_{10}, e_{13}] = e_9, & [e_{10}, e_{15}] = -ae_{10} + e_9 &
\end{array}
$$

For the generic case, the symmetry algebra comprises of a sixteen-dimensional semi direct product, $\mathbb{R}^6 \rtimes \mathbb{R}^{10}$. The nilradical and its complement are abelian spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$, respectively.

4.11.1 Subcase $a = 0$

The symmetries and nonzero brackets are, respectively,

$$
\begin{array}{ll}
e_1 = D_z, & e_2 = D_x, \\
e_3 = D_q, & e_4 = D_t, \\
e_5 = D_y, & e_6 = wD_t, \\
e_7 = \sin wD_q + \cos wD_x, & e_8 = \sin wD_x - \cos wD_q, \\
e_9 = e^{bw}(\sin cwD_y + \cos cwD_z), & e_{10} = e^{bw}(\cos cwD_y - \sin cwD_z), \\
e_{11} = tD_t, & e_{12} = qD_q + xD_x, \\
e_{13} = yD_y + zD_z, & e_{14} = zD_y - yD_z, \\
e_{15} = D_w + \frac{1}{2}(xD_q - qD_x), & e_{16} = qD_x - xD_q, \\
e_{17} = (x \sin w - q \cos w)D_q + (x \cos w + q \sin w)D_x, & \\
e_{18} = (x \cos w + q \sin w)D_q + (q \cos w - x \sin w)D_x. &
\end{array}
$$

$$
\begin{array}{llll}
[e_1, e_{13}] = e_1, & [e_1, e_{14}] = e_5, & [e_2, e_{12}] = e_2, & [e_2, e_{15}] = \frac{1}{2}e_3,
\end{array}
$$

$$[e_2, e_{16}] = -e_3, \quad [e_2, e_{17}] = e_7, \quad [e_2, e_{18}] = -e_8, \quad [e_3, e_{12}] = e_3,$$
$$[e_3, e_{15}] = -\frac{1}{2}e_2, \quad [e_3, e_{16}] = e_2, \quad [e_3, e_{17}] = e_8, \quad [e_3, e_{18}] = e_7,$$
$$[e_4, e_{11}] = e_4, \quad [e_5, e_{13}] = e_5, \quad [e_5, e_{14}] = -e_1, \quad [e_6, e_{11}] = e_6,$$
$$[e_6, e_{15}] = -e_4, \quad [e_7, e_{12}] = e_7, \quad [e_7, e_{15}] = \frac{1}{2}e_8, \quad [e_7, e_{16}] = e_8,$$
$$[e_7, e_{17}] = e_2, \quad [e_7, e_{18}] = e_3, \quad [e_8, e_{12}] = e_8, \quad [e_8, e_{15}] = -\frac{1}{2}e_7,$$
$$[e_8, e_{16}] = -e_7, \quad [e_8, e_{17}] = e_3, \quad [e_8, e_{18}] = -e_2, \quad [e_9, e_{13}] = e_9,$$
$$[e_9, e_{14}] = e_{10}, \quad [e_9, e_{15}] = -be_9 - ce_{10}, \quad [e_{10}, e_{13}] = e_{10}, \quad [e_{10}, e_{14}] = -e_9,$$
$$[e_{10}, e_{15}] = -be_{10} + ce_9, \quad [e_{16}, e_{17}] = 2e_{18}, \quad [e_{16}, e_{18}] = -2e_{17}, \quad [e_{17}, e_{18}] = -2e_{16}.$$

4.11.2 Subcase $b = 0$

The symmetries and nonzero brackets are, respectively,

$$e_1 = D_q, \qquad e_2 = D_y,$$
$$e_3 = D_z, \qquad e_4 = D_t,$$
$$e_5 = D_x, \qquad e_6 = wD_t,$$
$$e_7 = \sin cwD_y + \cos cwD_z, \qquad e_8 = \cos cwD_y - \sin cwD_z,$$
$$e_9 = e^{aw}(\sin wD_q + \cos wD_x), \qquad e_{10} = e^{aw}(\sin wD_x - \cos wD_q),$$
$$e_{11} = tD_t, \qquad e_{12} = yD_y + zD_z,$$
$$e_{13} = qD_q + xD_x, \qquad e_{14} = qD_x - xD_q,$$
$$e_{15} = D_w + \frac{c}{2}(zD_y - yD_z), \qquad e_{16} = zD_y - yD_z,$$
$$e_{17} = (z\cos cw + y\sin cw)D_y + (y\cos cw - z\sin cw)D_z,$$
$$e_{18} = (y\cos cw - z\sin cw)D_y - (z\cos cw + y\sin cw)D_z.$$

$$[e_1, e_{13}] = e_1, \quad [e_1, e_{14}] = e_5, \quad [e_2, e_{12}] = e_2, \quad [e_2, e_{15}] = -\frac{c}{2}e_3,$$

$$
\begin{array}{llll}
[e_2, e_{16}] = -e_3, & [e_2, e_{17}] = e_7, & [e_2, e_{18}] = e_8, & [e_3, e_{12}] = e_3, \\
[e_3, e_{15}] = \dfrac{c}{2}e_2, & [e_3, e_{16}] = e_2, & [e_3, e_{17}] = e_8, & [e_3, e_{18}] = -e_7, \\
[e_4, e_{11}] = e_4, & [e_5, e_{13}] = e_5, & [e_5, e_{14}] = -e_1, & [e_6, e_{11}] = e_6, \\
[e_6, e_{15}] = -e_4, & [e_7, e_{12}] = e_7, & [e_7, e_{15}] = -\dfrac{c}{2}e_8, & [e_7, e_{16}] = e_8, \\
[e_7, e_{17}] = e_2, & [e_7, e_{18}] = -e_3, & [e_8, e_{12}] = e_8, & [e_8, e_{15}] = \dfrac{c}{2}e_7, \\
[e_8, e_{16}] = -e_7, & [e_8, e_{17}] = e_3, & [e_8, e_{18}] = e_2, & [e_9, e_{13}] = e_9, \\
[e_9, e_{14}] = e_{10}, & [e_9, e_{15}] = -ae_9 + e_{10}, & [e_{10}, e_{13}] = e_{10}, & [e_{10}, e_{14}] = -e_9, \\
[e_{10}, e_{15}] = -ae_{10} - e_9, & [e_{16}, e_{17}] = -2e_{18}, & [e_{16}, e_{18}] = 2e_{17}, & [e_{17}, e_{18}] = 2e_{16}
\end{array}
$$

For both subcases, it is $\mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^5 \rtimes \mathbb{R}^{10})$ Levi decomposation algebra. The radical comprises of a five-dimensional abelain complement and a ten-dimensional abelain nilradical spanned by $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}$ and $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$, respectively. The $\mathfrak{sl}(2,\mathbb{R})$ is spanned by e_{16}, e_{17}, e_{18}.

4.11.3 Subcase $a = 0, b = 0$

The symmetries and nonzero brackets are, respectively,

$$
\begin{array}{ll}
e_1 = D_z, & e_2 = D_x, \\
e_3 = D_q, & e_4 = D_t, \\
e_5 = D_y, & e_6 = wD_t, \\
e_7 = \sin wD_q + \cos wD_x, & e_8 = -\cos wD_q + \sin wD_x, \\
e_9 = \sin cwqD_y + \cos cwD_z, & e_{10} = \cos cwD_y - \sin cwD_z, \\
e_{11} = tD_t, & e_{12} = qD_q + xD_x, \\
e_{13} = yD_y + zD_z, &
\end{array}
$$

$$e_{14} = D_w + \frac{1}{2}xD_q - \frac{1}{2}qD_x + \frac{c}{2}(zD_y - yD_z), \qquad e_{15} = qD_x - xD_q,$$

$$e_{16} = (x\cos w + q\sin w)D_q + (q\cos w - x\sin w)D_x,$$

$$e_{17} = (x\sin w - q\cos w)D_q + (x\cos w + q\sin w)D_x, \qquad e_{18} = zD_y - yD_z,$$

$$e_{19} = (z\cos cw + y\sin cw)D_y + (y\cos cw - z\sin cw)D_z,$$

$$e_{20} = (y\cos cw - z\sin cw)D_y - (z\cos cw + y\sin cw)D_z.$$

$$\begin{array}{llll}
[e_1, e_{13}] = e_1, & [e_1, e_{14}] = \frac{c}{2}e_5, & [e_1, e_{18}] = e_5, & [e_1, e_{19}] = e_{10}, \\
[e_1, e_{20}] = -e_9, & [e_2, e_{12}] = e_2, & [e_2, e_{14}] = \frac{1}{2}e_3, & [e_2, e_{15}] = -e_3, \\
[e_2, e_{16}] = -e_8, & [e_2, e_{17}] = e_7, & [e_3, e_{12}] = e_3, & [e_3, e_{14}] = -\frac{1}{2}e_2, \\
[e_3, e_{15}] = e_2, & [e_3, e_{16}] = e_7, & [e_3, e_{17}] = e_8, & [e_4, e_{11}] = e_4, \\
[e_5, e_{13}] = e_5, & [e_5, e_{14}] = -\frac{c}{2}e_1, & [e_5, e_{18}] = -e_1, & [e_5, e_{19}] = e_9, \\
[e_5, e_{20}] = e_{10}, & [e_6, e_{11}] = e_6, & [e_6, e_{14}] = -e_4, & [e_7, e_{12}] = e_7, \\
[e_7, e_{14}] = \frac{1}{2}e_8, & [e_7, e_{15}] = e_8, & [e_7, e_{16}] = e_3, & [e_7, e_{17}] = e_2, \\
[e_8, e_{12}] = e_8, & [e_8, e_{14}] = -\frac{1}{2}e_7, & [e_8, e_{15}] = -e_7, & [e_8, e_{16}] = -e_2, \\
[e_8, e_{17}] = e_3, & [e_9, e_{13}] = e_9, & [e_9, e_{14}] = -\frac{c}{2}e_{10}, & [e_9, e_{18}] = e_{10}, \\
[e_9, e_{19}] = e_5, & [e_9, e_{20}] = -e_1, & [e_{10}, e_{13}] = e_{10}, & [e_{10}, e_{14}] = \frac{c}{2}e_9, \\
[e_{10}, e_{18}] = -e_9, & [e_{10}, e_{19}] = e_1, & [e_{10}, e_{20}] = e_5, & [e_{15}, e_{16}] = -2e_{17}, \\
[e_{15}, e_{17}] = 2e_{16}, & [e_{16}, e_{17}] = 2e_{15}, & [e_{18}, e_{19}] = -2e_{20}, & [e_{18}, e_{20}] = 2e_{19}, \\
[e_{19}, e_{20}] = 2e_{18}. & & &
\end{array}$$

The symmetry algebra is indecomposable with Levi factor $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \rtimes (\mathbb{R}^4 \rtimes \mathbb{R}^{10})$, where the semisimple has a two copy of $\mathfrak{sl}(2,\mathbb{R})$ spanned by e_{15}, e_{16}, e_{17} and e_{18}, e_{19}, e_{20}. The nilradical $\mathbb{R}^{10}$ and its complement $\mathbb{R}^4$ are abelain spanned by $e_1, e_2, e_3,$

$e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and $e_{11}, e_{12}, e_{13}, e_{14}$, respectively.

4.11.4 Subcase $b = a$

The symmetries and nonzero brackets are, respectively,

$$
\begin{aligned}
&e_1 = D_y, && e_2 = D_z,\\
&e_3 = D_q, && e_4 = D_t,\\
&e_5 = D_x, && e_6 = wD_t,\\
&e_7 = e^{aw}(\sin wD_q + \cos wD_x), && e_8 = e^{aw}(-\cos wD_q + \sin wD_x),\\
&e_9 = e^{aw}(\sin cwD_y + \cos cwD_z), && e_{10} = e^{aw}\cos cwD_y - \sin cwD_z),\\
&e_{11} = tD_t, && e_{12} = D_w,\\
&e_{13} = yD_y + zD_z, && e_{14} = qD_q + xD_x,\\
&e_{15} = zD_y - yD_z, && e_{16} = qD_x - xD_q.
\end{aligned}
$$

$$
\begin{array}{llll}
[e_1, e_{13}] = e_1, & [e_1, e_{15}] = -e_2, & [e_2, e_{13}] = e_2, & [e_2, e_{15}] = e_1,\\
[e_3, e_{14}] = e_3, & [e_3, e_{16}] = e_5, & [e_4, e_{11}] = e_4, & [e_5, e_{14}] = e_5,\\
[e_5, e_{16}] = -e_3, & [e_6, e_{11}] = e_6, & [e_6, e_{12}] = -e_4, & [e_7, e_{12}] = -ae_7 + e_8,\\
[e_7, e_{14}] = e_7, & [e_7, e_{16}] = e_8, & [e_8, e_{12}] = -ae_8 - e_7, & [e_8, e_{14}] = e_8,\\
[e_8, e_{16}] = -e_7, & [e_9, e_{12}] = -ae_9 - ce_{10}, & [e_9, e_{13}] = e_9, & [e_9, e_{15}] = e_{10},\\
[e_{10}, e_{12}] = -ae_{10} + ce_9, & [e_{10}, e_{13}] = e_{10}, & [e_{10}, e_{15}] = -e_9. &
\end{array}
$$

The symmetry algebra is a sixteen-dimensional indecomposable solvable $\mathbb{R}^6 \rtimes \mathbb{R}^{10}$, where the nilradical and its complement are abelian spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}$ and $e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}$, respectively.

4.12 Algebra$^a_{5,18}$

The nonzero brackets are

$$[e_1,e_5]=ae_1-e_2,[e_2,e_5]=e_1+ae_2,[e_3,e_5]=e_1+ae_3-e_4,[e_4,e_5]=e_2+e_3+ae_4; \quad (a\geq 0), \tag{4.23}$$

and the corresponding system of geodesic equations is

$$\ddot{q}=a\dot{q}\dot{w}+\dot{x}\dot{w}+\dot{y}\dot{w}, \quad \ddot{x}=-\dot{q}\dot{w}+a\dot{x}\dot{w}+\dot{z}\dot{w}, \quad \ddot{y}=a\dot{y}\dot{w}+\dot{z}\dot{w}, \quad \ddot{z}=-\dot{y}\dot{w}+a\dot{z}\dot{w}, \quad \ddot{w}=0. \tag{4.24}$$

Symmetry algebra basis and nonvanishing brackets are, respectively,

$$\begin{aligned}
e_1 &= D_y,\\
e_2 &= D_x,\\
e_3 &= D_q,\\
e_4 &= D_t,\\
e_5 &= D_z,\\
e_6 &= wD_t,\\
e_7 &= yD_q+zD_x,\\
e_8 &= yD_x-zD_q+,\\
e_9 &= (\sin wa-\cos w)\frac{e^{aw}D_q}{(a^2+1)}+(a\cos w+\sin w)\frac{e^{aw}D_x}{(a^2+1)},\\
e_{10} &= (-\sin w-a\cos w)\frac{e^{aw}D_q}{(a^2+1)}+(\sin wa-\cos w)\frac{e^{aw}D_x}{(a^2+1)},\\
e_{11} &= (-w\cos w+\sin w)e^{aw}D_q+(w\sin w+\cos w)e^{aw}D_x+e^{aw}(-\cos wD_y+\sin wD_z),\\
e_{12} &= ((a^2w-2a+w)\sin w+2\cos w)\frac{e^{aw}D_q}{(a^2+1)}+((a^2w-2a+w)\cos w-2\sin w)\frac{e^{aw}D_x}{(a^2+1)}+\\
&\quad +e^{aw}(\sin wD_y+\cos wD_z),
\end{aligned}$$

$e_{13} = tD_t, \quad e_{14} = D_w,$

$e_{15} = qD_q + xD_x + yD_y + zD_z,$

$e_{16} = -xD_q + qD_x - zD_y + yD_z.$

$$
\begin{array}{lll}
[e_1, e_7] = e_3, & [e_1, e_8] = e_2, & [e_1, e_{15}] = e_1, \\
[e_1, e_{16}] = e_5, & [e_2, e_{15}] = e_2, & [e_2, e_{16}] = -e_3, \\
[e_3, e_{15}] = e_3, & [e_3, e_{16}] = e_2, & [e_4, e_{13}] = e_4, \\
[e_5, e_7] = e_2, & [e_5, e_8] = -e_3, & [e_5, e_{15}] = e_5, \\
[e_5, e_{16}] = -e_1, & [e_6, e_{13}] = e_6, & [e_6, e_{14}] = -e_4, \\
[e_7, e_{11}] = -ae_{10} - e_9, & [e_7, e_{12}] = -ae_9 + e_{10}, & [e_8, e_{11}] = ae_9 - e_{10}, \\
[e_8, e_{12}] = -ae_{10} - e_9, & [e_9, e_{14}] = -ae_9 + e_{10}, & [e_9, e_{15}] = e_9, \\
[e_9, e_{16}] = e_{10}, & [e_{10}, e_{14}] = -ae_{10} - e_9, & [e_{10}, e_{15}] = e_{10}, \\
[e_{10}, e_{16}] = -e_9, & [e_{11}, e_{14}] = -ae_{11} - e_{12} - 2e_9, & [e_{11}, e_{15}] = e_{11}, \\
[e_{11}, e_{16}] = ae_{10} - e_{12} - e_9, & [e_{12}, e_{14}] = -ae_{12} - 2ae_9 + e_{11}, & [e_{12}, e_{15}] = e_{12}, \\
[e_{12}, e_{16}] = -ae_9 - e_{10} + e_{11}. & &
\end{array}
$$

For the generic case, it is a sixteen-dimensional indecomposable solvable Lie algebra with a twelve-dimensional non-abelian nilradical spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$ and a four-dimensional abelain complement spanned by $e_{13}, e_{14}, e_{15}, e_{16}$.

4.12.1 Subcase $a = 0$

The symmetries and nonzero brackets are, respectively,

$e_1 = D_y,$

$e_2 = D_x,$

$$
\begin{aligned}
e_3 &= D_q, \\
e_4 &= D_t, \\
e_5 &= D_z, \\
e_6 &= wD_t, \\
e_7 &= yD_q + zD_x, \\
e_8 &= yD_x - zD_q, \\
e_9 &= \sin wD_x - \cos wD_q, \\
e_{10} &= -\sin wD_q - \cos wD_x, \\
e_{11} &= (z\sin w - y\cos w)D_q + (z\cos w + y\sin w)D_x, \\
e_{12} &= -(z\cos w + y\sin w)D_q + (z\sin w - y\cos w)D_x, \\
e_{13} &= (2\cos w + w\sin w)D_q + (w\cos w - 2\sin w)D_x + \sin wD_y + \cos wD_z, \\
e_{14} &= (\sin w - w\cos w)D_q + (\cos w + w\sin w)D_x - \cos wD_y + \sin wD_z, \\
e_{15} &= qD_q + xD_x + yD_y + zD_z, \\
e_{16} &= tD_t, \\
e_{17} &= D_w + \frac{1}{2}(xD_q - qD_x + zD_y - yD_z), \\
e_{18} &= qD_x - zD_y + yD_z - xD_q, \\
e_{19} &= (z\sin w - y\cos w)D_q + (z\cos w + y\sin w)D_x + \frac{1}{2}((wz - x + 2y)\cos w \\
&\quad - \sin w(q + 2z - wy))D_q + \frac{1}{2}((wy - q - 2z)\cos w - \sin w(wz - x + 2y))D_x \\
&\quad + \frac{1}{2}(z\cos w + y\sin w)D_y + \frac{1}{2}(y\cos w - z\sin w)D_z, \\
e_{20} &= -(z\cos w + y\sin w)D_q + (z\sin w - y\cos w)D_x + ((q + z - wy)\cos w \\
&\quad + (wz - x + y)\sin w)D_q + ((wz - x + y)\cos w - \sin w(q + z - wy))D_x \\
&\quad + (z\sin w - y\cos w)D_y + (z\cos w + y\sin w)D_z.
\end{aligned}
$$

$$
\begin{array}{lll}
[e_1,e_7]=e_3, & [e_1,e_8]=e_2, & [e_1,e_{11}]=e_9,\\
[e_1,e_{12}]=e_{10}, & [e_1,e_{15}]=e_1, & [e_1,e_{17}]=-\frac{1}{2}e_5,\\
[e_1,e_{18}]=e_5, & [e_1,e_{19}]=e_9+\frac{1}{2}e_{13}, & [e_1,e_{20}]=e_{10}+e_{14},\\
[e_2,e_{15}]=e_2, & [e_2,e_{17}]=\frac{1}{2}e_3, & [e_2,e_{18}]=-e_3,\\
[e_2,e_{19}]=\frac{1}{2}e_9, & [e_2,e_{20}]=e_{10}, & [e_3,e_{15}]=e_3,\\
[e_3,e_{17}]=-\frac{1}{2}e_2, & [e_3,e_{18}]=e_2, & [e_3,e_{19}]=\frac{1}{2}e_{10},\\
[e_3,e_{20}]=-e_9, & [e_4,e_{16}]=e_4, & [e_5,e_7]=e_2,\\
[e_5,e_8]=-e_3, & [e_5,e_{11}]=-e_{10}, & [e_5,e_{12}]=e_9,\\
[e_5,e_{15}]=e_5, & [e_5,e_{17}]=\frac{1}{2}e_1, & [e_5,e_{18}]=-e_1,\\
[e_5,e_{19}]=-\frac{1}{2}(e_{10}+e_{14}), & [e_5,e_{20}]=e_{13}+2e_9, & [e_6,e_{16}]=e_6,\\
[e_6,e_{17}]=-e_4, & [e_7,e_{13}]=e_{10}, & [e_7,e_{14}]=-e_9,\\
[e_7,e_{19}]=e_{12}, & [e_7,e_{20}]=-2e_{11}, & [e_8,e_{13}]=-e_9,\\
[e_8,e_{14}]=-e_{10}, & [e_9,e_{15}]=e_9, & [e_9,e_{17}]=\frac{1}{2}e_{10},\\
[e_9,e_{18}]=e_{10}, & [e_9,e_{19}]=\frac{1}{2}e_2, & [e_9,e_{20}]=-e_3,\\
[e_{10},e_{15}]=e_{10}, & [e_{10},e_{17}]=-\frac{1}{2}e_9, & [e_{10},e_{18}]=-e_9,\\
[e_{10},e_{19}]=\frac{1}{2}e_3, & [e_{10},e_{20}]=e_2, & [e_{11},e_{13}]=-e_2,\\
[e_{11},e_{14}]=-e_3, & [e_{11},e_{18}]=2e_{12}, & [e_{11},e_{20}]=-2e_7,\\
[e_{12},e_{13}]=e_3, & [e_{12},e_{14}]=-e_2, & [e_{12},e_{18}]=-2e_{11},\\
[e_{12},e_{19}]=e_7, & [e_{13},e_{15}]=e_{13}, & [e_{13},e_{17}]=\frac{1}{2}(e_{10}+e_{14}),\\
[e_{13},e_{18}]=-e_{10}+e_{14}, & [e_{13},e_{19}]=\frac{1}{2}e_1-e_2, & [e_{13},e_{20}]=2e_3+e_5,\\
[e_{14},e_{15}]=e_{14}, & [e_{14},e_{17}]=-\frac{1}{2}(e_{13}+3e_9), & [e_{14},e_{18}]=-e_{13}-e_9,
\end{array}
$$

$$[e_{14}, e_{19}] = -\frac{1}{2}(e_3 + e_5), \quad [e_{14}, e_{20}] = e_1 - e_2, \quad [e_{17}, e_{19}] = -\frac{1}{2}e_{12},$$
$$[e_{17}, e_{20}] = e_{11}, \quad [e_{18}, e_{19}] = -e_{20}, \quad [e_{18}, e_{20}] = 4e_{19},$$
$$[e_{19}, e_{20}] = e_{18}.$$

The symmetry algebra is $\mathfrak{sl}(2, \mathbb{R}) \ltimes (\mathbb{R}^3 \ltimes \mathbb{R}^{14})$ with nontrivial Levi decomposition. The semisimple part is $\mathfrak{sl}(2, \mathbb{R})$ spanned by e_{18}, e_{19}, e_{20}, whereas the nilradical $\mathbb{R}^{14}$ and its complement $\mathbb{R}^3$ are abelian spanned by $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}$ and e_{15}, e_{16}, e_{17}, respectively.

CHAPTER 5

LIE SYMMETRIES OF THE CANONICAL CONNECTION: CODIMENSION ONE ABELIAN NILRADICAL CASE

In this chapter, we construct the symmetries of the canonical connection for general Lie groups geodesic equations. We examine a particular class, namely, those groups for which the associated Lie algebras are solvable and have a codimension one abelian nilradical. Further, we show that the derived symmetries are coincided with the symmetries of the equation of geodesics of $A_{5,7}^{abc}$.

5.1 A Codimension One Abelian Nilradical Lie Algebra

Definition 5.1 ([36]). *Let $\mathfrak{g}$ be a Lie algebra with basis $\{e_1, \ldots, e_n\}$. $\mathfrak{g}$ has a codimension one abelian nilradical if the nonzero brackets of $\mathfrak{g}$ are linear combinations of*

$$[e_i, e_n] = \sum_{k=1}^{n-1} a_i^k e_k, \quad i \leq n-1. \tag{5.1}$$

To illustrate the definition, we consider the algebra $A_{5,7}^{abc}$ of Section 4.1, which is spanned by e_1, e_2, e_3, e_4, e_5 with the nonzero brackets

$$[e_1, e_5] = e_1,\ [e_2, e_5] = ae_2,\ [e_3, e_5] = be_3,\ [e_4, e_5] = ce_4;\ (abc \neq 0, -1 \leq c \leq b \leq a \leq 1). \tag{5.2}$$

Such an algebra is an example of a codimension one abelian nilradical, where the codimension is spanned by e_5.

Before continuing the analysis, we quote the following result, which has already been proved in [47]. The primary purpose of stating such a result is to provide us with a coordinate normal form for the canonical connection of a Lie algebra that has a codimension one abelian nilradical. Such Lie algebras are characterized by a matrix $\mathrm{ad}(e_{n+1})$ where e_{n+1} is a fixed element of $\mathfrak{g}$, usually taken as the last element of a basis.

Theorem 5.1. (i) *Suppose that $\mathfrak{g}$ is Lie algebra of dimension $n+1$ such that there exists a basis $\{e_1, e_2, ..., e_n, e_{n+1}\}$ for which the only non-zero brackets in $\mathfrak{g}$ are given by $[e_i, e_{n+1}] = a_i^j e_j$, where, on the right hand side the summation over j extends from 1 to n. Then there exists a coordinate system (x^i, w) on the local Lie group G associated to $\mathfrak{g}$, such that $\mathfrak{g}$ is faithfully represented by (X_i, W) where $X_i = \frac{\partial}{\partial x^i}$ and $W = \frac{\partial}{\partial w} + a_j^k x^j \frac{\partial}{\partial x^k}$.*
(ii) *In the coordinate system (x^i, w) of (i) the geodesic equations of the canonical connection are given by*

$$\ddot{x}^i = a_j^i \dot{x}^j \dot{w}, \quad \ddot{w} = 0. \tag{5.3}$$

5.2 Lie Symmetries of the Canonical Connection

We write the system of geodesic equations (1.1), that is,

$$\ddot{x}^i = -\Gamma^i_{jk} \dot{x}^j \dot{x}^k, \quad (i, j, k = 1, \ldots, n), \tag{5.4}$$

as

$$\dot{u}^i = f^i(t, x^j, u^j), \tag{5.5}$$

where $f^i(t, x^j, u^j) = -\Gamma^i_{jk} u^j u^k$ and u^i denotes the time derivative of x^i. We encode $\dot{u}^i = f^i(t, x^j, u^j)$ in terms of the associated differential linear operator

$$\Gamma = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + f^i(t, x^j, u^j) \frac{\partial}{\partial u^i}. \tag{5.6}$$

In order to find the condition for a Lie symmetry for (5.5), we need a vector field of the form $\mathbf{X} = \xi(t,x)\frac{\partial}{\partial t} + \eta^i(t,x)\frac{\partial}{\partial x^i}$ such that $\left[\tilde{\mathbf{X}}, \Gamma\right] = \lambda(t,x,u^j)\Gamma$ for some function λ, where $\tilde{\mathbf{X}}$ is the first prolongation of $\mathbf{X}$ given by

$$\tilde{\mathbf{X}} = \xi(t,x,u^i)\frac{\partial}{\partial t} + \eta^i(t,x,u^i)\frac{\partial}{\partial x^i} + P^i\frac{\partial}{\partial u^i}, \tag{5.7}$$

$P^i = \dot{\eta}^i - u^i\dot{\xi} \implies \dot{P}^i = \ddot{\eta}^i - u^i\ddot{\xi} - f^i\dot{\xi}$.

5.3 Determination of all Lie Symmetries

The Lie symmetry condition for the system of geodesic equations (5.5) is

$$\left[\tilde{\mathbf{X}}, \Gamma\right] - \lambda\Gamma = 0, \tag{5.8}$$

$$[\xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial x^i} + P^i\frac{\partial}{\partial u^i}, \Gamma] - \lambda\Gamma = 0, \tag{5.9}$$

$$\left[\xi\frac{\partial}{\partial t}, \Gamma\right] + \left[\eta^i\frac{\partial}{\partial x^i}, \Gamma\right] + \left[P^i\frac{\partial}{\partial u^i}, \Gamma\right] - \lambda\Gamma = 0, \tag{5.10}$$

$$\xi\left[\frac{\partial}{\partial t}, \Gamma\right] - \dot{\xi}\frac{\partial}{\partial t} - \dot{\eta}^i\frac{\partial}{\partial x^i} + \eta^i\left[\frac{\partial}{\partial x^i}, \Gamma\right] - \dot{P}^i\frac{\partial}{\partial u^i} + P^i\left[\frac{\partial}{\partial u^i}, \Gamma\right] - \lambda\Gamma = 0, \tag{5.11}$$

$$\begin{aligned}\xi\frac{\partial f^i}{\partial t}\frac{\partial}{\partial u^i} - \dot{\xi}\frac{\partial}{\partial t} - \dot{\eta}^i\frac{\partial}{\partial x^i} + \eta^j\frac{\partial f^i}{\partial x^j}\frac{\partial}{\partial u^i} - \dot{P}^i\frac{\partial}{\partial u^i} + P^i\Big(\frac{\partial}{\partial x^i} + \frac{\partial f^j}{\partial u^i}\frac{\partial}{\partial u^j}\Big) \\ - \lambda\Big(\frac{\partial}{\partial t} + u^i\frac{\partial}{\partial x^i} + f^i\frac{\partial}{\partial u^i}\Big) = 0,\end{aligned} \tag{5.12}$$

$$-\Big(\dot{\xi}+\lambda\Big)\frac{\partial}{\partial t} + \Big(P^i - \dot{\eta}^i - \lambda u^i\Big)\frac{\partial}{\partial x^i} + \Big(\xi\frac{\partial f^i}{\partial t} + \eta^j\frac{\partial f^i}{\partial x^j} - \dot{P}^i + P^j\frac{\partial f^i}{\partial u^j} - \lambda f^i\Big)\frac{\partial}{\partial u^i} = 0. \tag{5.13}$$

Equating to zero the coefficients of $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x^i}$, and $\frac{\partial}{\partial u^i}$ leads to the following three symmetry conditions:

$$-\lambda - \dot{\xi} = 0 \implies \lambda = -\dot{\xi}, \tag{5.14}$$

$$P^i - \dot{\eta}^i - \lambda u^i = \dot{\eta}^i - u^i\dot{\xi} - \dot{\eta}^i + u^i\dot{\xi} \equiv 0, \tag{5.15}$$

$$\xi\frac{\partial f^i}{\partial t} + \eta^j\frac{\partial f^i}{\partial x^j} - \dot{P}^i + P^j\frac{\partial f^i}{\partial u^j} - \lambda f^i = 0. \tag{5.16}$$

However,

$$f^i(t, x^j, u^j) = -\Gamma^i_{jk} u^j u^k \quad \text{so} \quad \frac{\partial f^i}{\partial t} = 0. \tag{5.17}$$

Thus,

$$\eta^j \frac{\partial f^i}{\partial x^j} - \dot{P}^i + P^j \frac{\partial f^i}{\partial u^j} + f^i \dot{\xi} = 0, \tag{5.18}$$

$$\eta^j \frac{\partial f^i}{\partial x^j} - \left(\ddot{\eta}^i - u^i \ddot{\xi} - f^i \dot{\xi}\right) + \left(\dot{\eta}^j - u^j \dot{\xi}\right) \frac{\partial f^i}{\partial u^j} + f^i \dot{\xi} = 0, \tag{5.19}$$

$$\eta^j \frac{\partial f^i}{\partial x^j} - \ddot{\eta}^i + u^i \ddot{\xi} + f^i \dot{\xi} + \dot{\eta}^j \frac{\partial f^i}{\partial u^j} - \underbrace{u^j \frac{\partial f^i}{\partial u^j}}_{2f^i} \dot{\xi} + f^i \dot{\xi} = 0, \tag{5.20}$$

$$\underbrace{\eta^j \frac{\partial f^i}{\partial x^j}}_{j\,=\,i} - \ddot{\eta}^i + u^i \ddot{\xi} + \dot{\eta}^j \frac{\partial f^i}{\partial u^j} = 0, \tag{5.21}$$

$$\ddot{\eta}^i - u^i \ddot{\xi} - \dot{\eta}^j \frac{\partial f^i}{\partial u^j} - \eta^i \frac{\partial f^j}{\partial x^i} = 0, \tag{5.22}$$

$\xi = \xi(t, x^i)$ and $\eta^i = \eta^i(t, x^i)$; thus

$$\dot{\xi} = \frac{\partial}{\partial t}(\xi) + u^i \frac{\partial}{\partial x^i}(\xi) = \xi_t + u^i \xi_{x^i}, \tag{5.23}$$

$$\eta^i = \frac{\partial}{\partial t}(\eta^i) + u^j \frac{\partial}{\partial x^j}(\eta^i) = \eta^i_t + u^j \eta^i_{x^j}, \tag{5.24}$$

which imply to the following identities:

$$\ddot{\xi} = \xi_{tt} + 2u^j \xi_{tx^j} + u^j u^k \xi_{x^j x^k} + f^i \xi_{x^i}, \tag{5.25}$$

$$\ddot{\eta}^i = \eta^i_{tt} + 2u^j \eta^i_{tx^j} + u^j u^k \eta^i_{x^j x^k} + f^j \eta^i_{x^j}. \tag{5.26}$$

Next, we shall make use of the identities, equations (5.25) and (5.26), where we have used subscripts to denote derivatives. Thus, equation (5.22) becomes

$$\left(\eta^i_{tt} + 2u^j \eta^i_{tx^j} + u^j u^k \eta^i_{x^j x^k} + f^j \eta^i_{x^j}\right) - u^i \left(\xi_{tt} + 2u^j \xi_{tx^j} + u^j u^k \xi_{x^j x^k} + f^i \xi_{x^i}\right)$$
$$- \left(\eta^j_t + u^k \eta^j_{x^k}\right) \frac{\partial f^i}{\partial u^j} - \eta^i \frac{\partial f^j}{\partial x^i} = 0. \tag{5.27}$$

Equation (5.27) contains terms of degree zero, one, two and three in the u^i and we write down the coefficients of each term so as to obtain

$$\frac{\partial^2 \eta^i}{\partial t^2} = 0, \tag{5.28}$$

$$2u^j \frac{\partial^2 \eta^i}{\partial x^j \partial t} - u^i \frac{\partial^2 \xi}{\partial t^2} - \frac{\partial \eta^j}{\partial t} \frac{\partial f^i}{\partial u^j} = 0, \tag{5.29}$$

$$u^j u^k \frac{\partial^2 \eta^i}{\partial x^j \partial x^k} + f^j \eta^i_{x^j} - \eta^j \frac{\partial f^i}{\partial x^j} - 2u^i u^j \frac{\partial^2 \xi}{\partial t \partial x^j} - u^k \frac{\partial \eta^j}{\partial x^k} \frac{\partial f^i}{\partial u^j} = 0, \tag{5.30}$$

$$u^i u^j u^k \frac{\partial^2 \xi}{\partial x^j \partial x^k} + u^i f^k \frac{\partial \xi}{\partial x^k} = 0. \tag{5.31}$$

Using equation (5.17), then equations (5.29), (5.30), and (5.31) give, on equating powers of the u^i to zero,

$$2\frac{\partial^2 \eta^i}{\partial x^k \partial t} - \delta^i_k \frac{\partial^2 \xi}{\partial t^2} + 2\frac{\partial \eta^j}{\partial t} \Gamma^i_{jk} = 0, \tag{5.32}$$

$$\frac{\partial^2 \eta^i}{\partial x^k \partial x^m} + \frac{\partial \eta^j}{\partial x^k} \Gamma^i_{jm} + \frac{\partial \eta^j}{\partial x^m} \Gamma^i_{jk} + \eta^j \frac{\partial \Gamma^i_{km}}{\partial x^j} - \delta^i_k \frac{\partial^2 \xi}{\partial t \partial x^m} - \delta^i_m \frac{\partial^2 \xi}{\partial t \partial x^k} - \eta^i_{x^j} \Gamma^j_{km} = 0, \tag{5.33}$$

$$\frac{\partial^2 \xi}{\partial x^j \partial x^k} - \Gamma^i_{jk} \xi_{x^i} = 0. \tag{5.34}$$

5.4 Codimension One Abelian Nilradical Case

Next, we shall adapt equations (5.28), (5.32), (5.33), and (5.34) to the codimension one abelian nilradical case. Note that the Lie algebra is now of dimension $n+1$. The first n coordinates are denoted by x^i and the $(n+1)$th by w. We shall denote a Lie symmetry now by

$$\mathbf{X} = \xi(t, x^i, w)\frac{\partial}{\partial t} + \eta^i(t, x^i, w)\frac{\partial}{\partial x^i} + \eta(t, x^i, w)\frac{\partial}{\partial w}. \tag{5.35}$$

We shall employ the summation convention in the range 1 to n. The connection components coming from equation (5.3) are given by

$$\left.\begin{array}{l}\Gamma^i_{jk}=0, \quad \Gamma^i_{jn+1}=\Gamma^i_{n+1j}=-\frac{1}{2}a^i_j, \quad \Gamma^i_{n+1n+1}=0,\\ \Gamma^{n+1}_{ij}=0, \quad \Gamma^{n+1}_{jn+1}=\Gamma^{n+1}_{n+1j}=0, \quad \Gamma^{n+1}_{n+1n+1}=0.\end{array}\right\} \tag{5.36}$$

We find the following conditions on ξ, η^i, η where in the interest of brevity, derivatives with respect to t and w are denoted by subscripts and derivatives with respect to x^i is denoted by subscript i and $1 \le i, j, k, m \le n$. Hence, we have the following system of PDE's:

$$\xi_{x^jx^k}=0, \tag{5.37}$$

$$\xi_{x^iw}+\frac{1}{2}a^j_i\xi_{x^j}=0, \tag{5.38}$$

$$\xi_{ww}=0, \tag{5.39}$$

$$\eta_{tt}=0, \tag{5.40}$$

$$\eta_{tx^k}=0, \tag{5.41}$$

$$\eta_{x^kx^m}=0, \tag{5.42}$$

$$\eta_{x^kw}-\xi_{tx^k}+\frac{1}{2}a^j_k\eta_{x^j}=0, \tag{5.43}$$

$$\eta_{ww}-2\xi_{tw}=0, \tag{5.44}$$

$$2\eta_{tw}-\xi_{tt}=0, \tag{5.45}$$

$$\eta^i_{tt}=0, \tag{5.46}$$

$$\eta^i_{tw}-\frac{1}{2}a^i_k\eta^k_t=0, \tag{5.47}$$

$$\eta^i_{ww}-a^i_j\eta^j_w=0, \tag{5.48}$$

$$2\eta^i_{tx^k}-\delta^i_k\xi_{tt}-a^i_k\eta_t=0, \tag{5.49}$$

$$\eta^i_{x^kx^m}-\frac{1}{2}a^i_m\eta_{x^k}-\frac{1}{2}a^i_k\eta_{x^m}-\delta^i_k\xi_{tx^m}-\delta^i_m\xi_{tx^k}=0, \tag{5.50}$$

$$\eta^i_{x^k w} - \frac{1}{2}a^i_j \eta^j_{x^k} - \frac{1}{2}a^i_k \eta_w + \frac{1}{2}a^j_k \eta^i_{x^j} - \delta^i_k \xi_{tw} = 0. \tag{5.51}$$

5.5 Solving the PDE's System

Now, we integrate the system of PDE's to the extent possible. As such the solution to equations (5.40), (5.41), and (5.42) is given by

$$\eta = B_k(w)x^k + C(w)t + D(w). \tag{5.52}$$

Turning now to (5.38) and (5.39). If we take the w-derivative of (5.38) and use (5.38), we conclude that

$$a^j_i \xi_{x^j w} = 0. \tag{5.53}$$

Let us continue by assuming that the matrix A is non-singular. Then $\xi_{x^j w} = 0$ and using (5.38) again, we find that $\xi_{x^j} = 0$. As such, from (5.39) we conclude that

$$\xi = E(t)w + F(t). \tag{5.54}$$

Now from (5.44) and (5.45) we obtain $\xi_{ttw} = 0$ and $\eta_{tww} = 0$. From the w-derivative of (5.43) and the x^k-derivative of (5.44) we deduce that $\eta_{x^j w} = 0$ and hence again from (5.43) that $\eta_{x^j} = 0$. Integrating (5.44) and (5.45) gives

$$\xi = (Ft + G)w + K + Lt \quad \text{and} \quad \eta = Ct + Fw^2 + Hw + J. \tag{5.55}$$

Now we consider (5.46),(5.47) and (5.48). Take the w-derivative of (5.47) and the t-derivative of (5.48). We deduce that $\eta^k_{tw} = 0$ and hence from (5.47) that $\eta^k_t = 0$, since we are assuming that A is non-singular. Concerning (5.48) we find that the solution is

$$\eta^i = A^{-1}e^{wA}M^i(x) + N^i(x). \tag{5.56}$$

It remains to examine (5.49), (5.50) and (5.51). However, we see that in view of the conditions that have already been solved, (5.49) is satisfied identically. Furthermore, (5.50) reduces to

$$\eta^i_{x^k x^m} = 0. \tag{5.57}$$

Hence, we may write

$$\eta^i = (a^{-1})^i_j e^{wa^j_k}(P^k_m x^m + Q^k) + R^i_j x^j + S^i. \tag{5.58}$$

At this point, only (5.51) remains to be satisfied and it is

$$e^{wa^i_j} P^j_k + a^j_k((a^{-1})^i_m e^{wa^m_l} P^l_j + R^i_j) - a^i_j((a^{-1})^j_m e^{wa^m_l} P^l_k + R^j_k) - a^i_k(2Fw + H) - \delta^i_k F = 0. \tag{5.59}$$

Equation (5.59) splits into the following two conditions:

$$a^j_k (a^{-1})^i_m e^{wa^m_l} P^l_j = 0, \tag{5.60}$$

and

$$a^j_k R^i_j - a^i_j R^j_k - (2Fw + H)a^i_k - F\delta^i_k = 0. \tag{5.61}$$

Note that there are no conditions on Q^k and S^k and that necessarily $P^l_j = 0$ and $F = 0$. After putting $F = 0$, equation (5.61) may be written in matrix form as

$$[R, A] = HA, \tag{5.62}$$

where the left-hand side is a commutator. To summarize, the solution for ξ, η^i, η is given by equation (5.55) with $F = 0$ and

$$\eta^i = R^i_j x^j + S^i + e^{wa^i_k} T^k, \tag{5.63}$$

and H and R satisfy condition (5.62). Hence, the analytical solutions of the PDE's system lead to the following result.

Theorem 5.2. *The symmetries for the geodesic equations of the canonical connection (5.3), that is,* $\ddot{x}^i = a^i_j \dot{x}^j \dot{w}$, $\ddot{w} = 0$, *associated to the class of* $(n+1)$*-dimensional Lie algebras that have a codimension one abelian nilradical are given by (5.35), that is,* $\mathbf{X} = \xi(t, x^i, w)\frac{\partial}{\partial t} + \eta^i(t, x^i, w)\frac{\partial}{\partial x^i} + \eta(t, x^i, w)\frac{\partial}{\partial w}$, *where*

$$\begin{cases} \xi(t, x^i, w) = Lt + Gw + K, \\ \eta(t, x^i, w) = Ct + Hw + J, \\ \eta^i(t, x^i, w) = R^i_j x^j + e^{wa^i_k} T^k + S^i. \end{cases} \tag{5.64}$$

5.6 Comparison of Symmetries

In this section, following the previous analytical derivation of symmetries, we will compare with the symmetries obtained by MAPLE for the geodesics of $A^{abc}_{5,7}$. First, we examine whether the matrix A for $A^{abc}_{5,7}$ is singular or not. The geodesic equations are given

$$\ddot{x}^i = -\Gamma^i_{jk} \dot{x}^j \dot{x}^k, \quad (i = 1, \ldots, n), \tag{5.65}$$

and of the canonical connection associated with $(n+1)$-dimensional Lie algebra are given by

$$\ddot{x}^i = a^i_j \dot{x}^j \dot{w}, \quad \ddot{w} = 0, \quad (i = 1, \ldots, n). \tag{5.66}$$

Since the geodesic system of $A^{abc}_{5,7}$ is of dimension five, (5.65) becomes

$$\ddot{x}^i = -\Gamma^i_{jk} \dot{x}^j \dot{x}^k, \quad (i = 1, \ldots, 5), \text{ (sum over } j, k = 1, \ldots, 5). \tag{5.67}$$

The coordinates are now (x^1, x^2, x^3, x^4, w), which stand for (q, x, y, z, w). Hence,

$$i = 1 \begin{cases} \ddot{x}^1 = & -\Gamma^1_{11}\dot{x}^1\dot{x}^1 - \Gamma^1_{12}\dot{x}^1\dot{x}^2 - \Gamma^1_{13}\dot{x}^1\dot{x}^3 - \Gamma^1_{14}\dot{x}^1\dot{x}^4 - \Gamma^1_{15}\dot{x}^1\dot{x}^5 \\ & -\Gamma^1_{21}\dot{x}^2\dot{x}^1 - \Gamma^1_{22}\dot{x}^2\dot{x}^2 - \Gamma^1_{23}\dot{x}^2\dot{x}^3 - \Gamma^1_{24}\dot{x}^2\dot{x}^4 - \Gamma^1_{25}\dot{x}^2\dot{x}^5 \\ & -\Gamma^1_{31}\dot{x}^3\dot{x}^1 - \Gamma^1_{32}\dot{x}^3\dot{x}^2 - \Gamma^1_{33}\dot{x}^3\dot{x}^3 - \Gamma^1_{34}\dot{x}^3\dot{x}^4 - \Gamma^1_{35}\dot{x}^3\dot{x}^5 \\ & -\Gamma^1_{41}\dot{x}^4\dot{x}^1 - \Gamma^1_{42}\dot{x}^4\dot{x}^2 - \Gamma^1_{43}\dot{x}^4\dot{x}^3 - \Gamma^1_{44}\dot{x}^4\dot{x}^4 - \Gamma^1_{45}\dot{x}^4\dot{x}^5 \\ & -\Gamma^1_{51}\dot{x}^5\dot{x}^1 - \Gamma^1_{52}\dot{x}^5\dot{x}^2 - \Gamma^1_{53}\dot{x}^5\dot{x}^3 - \Gamma^1_{54}\dot{x}^5\dot{x}^4 - \Gamma^1_{55}\dot{x}^5\dot{x}^5 \end{cases} \tag{5.68}$$

$$\Longrightarrow \begin{cases} \ddot{q} = & -\Gamma^1_{11}\dot{q}\dot{q} - \Gamma^1_{12}\dot{q}\dot{x} - \Gamma^1_{13}\dot{q}\dot{y} - \Gamma^1_{14}\dot{q}\dot{z} - \Gamma^1_{15}\dot{q}\dot{w} \\ & -\Gamma^1_{21}\dot{x}\dot{q} - \Gamma^1_{22}\dot{x}\dot{x} - \Gamma^1_{23}\dot{x}\dot{y} - \Gamma^1_{24}\dot{x}\dot{z} - \Gamma^1_{25}\dot{x}\dot{w} \\ & -\Gamma^1_{31}\dot{y}\dot{q} - \Gamma^1_{32}\dot{y}\dot{x} - \Gamma^1_{33}\dot{y}\dot{y} - \Gamma^1_{34}\dot{y}\dot{z} - \Gamma^1_{35}\dot{y}\dot{w} \\ & -\Gamma^1_{41}\dot{z}\dot{q} - \Gamma^1_{42}\dot{z}\dot{x} - \Gamma^1_{43}\dot{z}\dot{y} - \Gamma^1_{44}\dot{z}\dot{z} - \Gamma^1_{45}\dot{z}\dot{w} \\ & -\Gamma^1_{51}\dot{w}\dot{q} - \Gamma^1_{52}\dot{w}\dot{x} - \Gamma^1_{53}\dot{w}\dot{y} - \Gamma^1_{54}\dot{w}\dot{z} - \Gamma^1_{55}\dot{w}\dot{w}. \end{cases} \tag{5.69}$$

Recall that the geodesics of $A^{abc}_{5,7}$, (4.2), are $\ddot{q} = \dot{q}\dot{w}$, $\ddot{x} = a\dot{x}\dot{w}$, $\ddot{y} = b\dot{y}\dot{w}$, $\ddot{z} = c\dot{z}\dot{w}$, $\ddot{w} = 0$. Hence, $\ddot{q} = -2\Gamma^1_{15}\dot{q}\dot{w} \implies \Gamma^1_{15} = -\frac{1}{2}$, and therefore

$$\Gamma^1_{jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.70}$$

Analogously, for $i = 2, 3, 4, 5$, we obtain

$$\Gamma^2_{jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{a}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{a}{2} & 0 & 0 & 0 \end{bmatrix}, \; \Gamma^3_{jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{b}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{b}{2} & 0 & 0 \end{bmatrix}, \; \Gamma^4_{jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{c}{2} \\ 0 & 0 & 0 & -\frac{c}{2} & 0 \end{bmatrix},$$

$$\Gamma^5_{jk} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.71}$$

Furthermore, the connection components (5.36) in the range 1 to 4 are

$$\begin{cases} \Gamma^i_{jk} = 0, \quad (i, j, k = 1, \dots, 4), \\ \Gamma^i_{j5} = \Gamma^i_{5j} = -\frac{1}{2} a^i_j, \quad (i, j = 1, \dots, 4), \\ \Gamma^i_{55} = 0, \quad (i = 1, \dots, 4), \\ \Gamma^5_{ij} = 0, \quad \Gamma^5_{i5} = \Gamma^5_{5j} = 0, \quad \Gamma^5_{55} = 0 \quad (i, j = 1 \dots, 4). \end{cases} \tag{5.72}$$

The only remaining quantities, $\Gamma^i_{j5} = \Gamma^i_{5j} = -\frac{1}{2} a^i_j$, can be computed as follows.

$$i = 1 \begin{cases} \Gamma^1_{15} = -\frac{1}{2} \implies -\frac{1}{2} a^1_1 = -\frac{1}{2} \implies a^1_1 = 1, \\ \Gamma^1_{25} = 0 \implies -\frac{1}{2} a^1_2 = 0 \implies a^1_2 = 0, \\ \Gamma^1_{35} = 0 \implies -\frac{1}{2} a^1_3 = 0 \implies a^1_3 = 0, \\ \Gamma^1_{45} = 0 \implies -\frac{1}{2} a^1_4 = 0 \implies a^1_4 = 0. \end{cases} \tag{5.73}$$

$$i=2\begin{cases}\Gamma^2_{15}=0 \implies -\frac{1}{2}a^2_1=0 \implies a^2_1=0,\\ \Gamma^2_{25}=-\frac{a}{2} \implies -\frac{1}{2}a^2_2=-\frac{a}{2} \implies a^2_2=a,\\ \Gamma^2_{35}=0 \implies -\frac{1}{2}a^2_3=0 \implies a^2_3=0,\\ \Gamma^2_{45}=0 \implies -\frac{1}{2}a^2_4=0 \implies a^2_4=0.\end{cases} \tag{5.74}$$

$$i=3\begin{cases}\Gamma^3_{15}=0 \implies -\frac{1}{2}a^3_1=0 \implies a^3_1=0,\\ \Gamma^3_{25}=0 \implies -\frac{1}{2}a^3_2=0 \implies a^3_2=0,\\ \Gamma^3_{35}=-\frac{b}{2} \implies -\frac{1}{2}a^3_3=-\frac{b}{2} \implies a^3_3=b,\\ \Gamma^3_{45}=0 \implies -\frac{1}{2}a^3_4=0 \implies a^3_4=0.\end{cases} \tag{5.75}$$

$$i=4\begin{cases}\Gamma^4_{15}=0 \implies -\frac{1}{2}a^4_1=0 \implies a^4_1=0,\\ \Gamma^4_{25}=0 \implies -\frac{1}{2}a^4_2=0 \implies a^4_2=0,\\ \Gamma^4_{35}=0 \implies -\frac{1}{2}a^4_3=0 \implies a^4_3=0,\\ \Gamma^4_{45}=-\frac{c}{2} \implies -\frac{1}{2}a^4_4=-\frac{c}{2} \implies a^4_4=c.\end{cases} \tag{5.76}$$

Hence, the matrix A corresponding to the geodesics of $A^{abc}_{5,7}$ is

$$A=\begin{bmatrix}1&0&0&0\\0&a&0&0\\0&0&b&0\\0&0&0&c\end{bmatrix} \underset{\times(-1)}{\Longrightarrow} A=\begin{bmatrix}-1&0&0&0\\0&-a&0&0\\0&0&-b&0\\0&0&0&-c\end{bmatrix}, \tag{5.77}$$

and clearly $det(A)\neq 0$, that is, $A^{abc}_{5,7}$ is non-singular.

Next, we verify that the symmetry vector fields for the geodesics of $A^{abc}_{5,7}$ exist, meaning that they match with the analytic derived symmetries (5.64). Let the symmetry generator (5.35) be

$$\mathbf{X}_r = \xi(t, x^i, w)\frac{\partial}{\partial t} + \eta^i(t, x^i, w)\frac{\partial}{\partial x^i} + \eta(t, x^i, w)\frac{\partial}{\partial w}, \ \ (i = 1, .., 4)$$

$$\mathbf{X}_r = \xi\frac{\partial}{\partial t} + \eta^1\frac{\partial}{\partial x^1} + \eta^2\frac{\partial}{\partial x^2} + \eta^3\frac{\partial}{\partial x^3} + \eta^4\frac{\partial}{\partial x^4} + \eta\frac{\partial}{\partial w} \iff$$

$$\iff \mathbf{X}_r = \xi\frac{\partial}{\partial t} + \eta^1\frac{\partial}{\partial q} + \eta^2\frac{\partial}{\partial x} + \eta^3\frac{\partial}{\partial y} + \eta^4\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial w}, \ (r = 1, \ldots, 16)$$

Recall that from equations (5.64), we have

$$\begin{cases} \xi = Lt + Gw + K, \\ \eta = Ct + Hw + J, \\ \eta^i = R^i_j x^j + e^{wa^i_k}T^k + S^i, \end{cases} \tag{5.78}$$

and the first symmetry vector field in the generic case of $A^{abc}_{5,7}$ is

$$e_1 = \frac{\partial}{\partial z}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i, j, k = 1, 2, 3), \\ \eta^4 = R^4_1 x^1 + R^4_2 x^2 + R^4_3 x^3 + R^4_4 x^4 + S^4 + e^{wa^4_1}T^1 + e^{wa^4_2}T^2 + e^{wa^4_3}T^3 + e^{wa^4_4}T^4, \\ \eta^4 = R^4_1 q + R^4_2 x + R^4_3 y + R^4_4 z + S^4 + e^{wa^4_1}T^1 + e^{wa^4_2}T^2 + e^{wa^4_3}T^3 + e^{wa^4_4}T^4, \\ \eta^4 = 1 \implies S^4 = 1 \text{ and } R^4_1 = R^4_2 = R^4_3 = R^4_4 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_1 = \eta^4\frac{\partial}{\partial z} \implies \mathbf{X}_1 = \frac{\partial}{\partial z}. \text{ Hence, } e_1 = \mathbf{X}_1. \end{cases}$$

In an analogous manner,

$$e_2 = \frac{\partial}{\partial w}\begin{cases} \xi = 0, \quad \eta^i = 0, \quad (i = 1, .., 4), \\ \eta = 1 \implies J = 1 \text{ and } C = H = 0, \\ \therefore \mathbf{X}_2 = \eta\frac{\partial}{\partial w} \implies \mathbf{X}_2 = \frac{\partial}{\partial w}. \text{ Hence, } e_2 = \mathbf{X}_2. \end{cases}$$

$$e_3 = t\frac{\partial}{\partial t}\begin{cases} \eta = 0, \quad \eta^i = 0, \quad (i = 1, .., 4), \\ \xi = t \implies L = 1 \text{ and } G = K = 0, \\ \therefore \mathbf{X}_3 = \xi\frac{\partial}{\partial t} \implies \mathbf{X}_3 = t\frac{\partial}{\partial t}. \text{ Hence, } e_3 = \mathbf{X}_3. \end{cases}$$

$$e_4 = \frac{\partial}{\partial t}\begin{cases} \eta = 0, \quad \eta^i = 0, \quad (i = 1, .., 4), \\ \xi = 1 \implies K = 1 \text{ and } G = L = 0, \\ \therefore \mathbf{X}_4 = \xi\frac{\partial}{\partial t} \implies \mathbf{X}_4 = \frac{\partial}{\partial t}. \text{ Hence, } e_4 = \mathbf{X}_4. \end{cases}$$

$$e_5 = \frac{\partial}{\partial q}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i, j, k = 2, 3, 4), \\ \eta^1 = R^1_1 q + R^1_2 x + R^1_3 y + R^1_4 z + S^1 + e^{wa^1_1}T^1 + e^{wa^1_2}T^2 + e^{wa^1_3}T^3 + e^{wa^1_4}T^4, \\ \eta^1 = 1 \implies S^1 = 1 \text{ and } R^1_1 = R^1_2 = R^1_3 = R^1_4 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_5 = \eta^1\frac{\partial}{\partial q} \implies \mathbf{X}_5 = \frac{\partial}{\partial q}. \text{ Hence, } e_5 = \mathbf{X}_5. \end{cases}$$

$$e_6 = \frac{\partial}{\partial x}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i, j, k = 1, 3, 4), \\ \eta^2 = R^2_1 q + R^2_2 x + R^2_3 y + R^2_4 z + S^2 + e^{wa^2_1}T^1 + e^{wa^2_2}T^2 + e^{wa^2_3}T^3 + e^{wa^2_4}T^4, \\ \eta^2 = 1 \implies S^2 = 1 \text{ and } R^2_1 = R^2_2 = R^2_3 = R^2_4 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_6 = \eta^2\frac{\partial}{\partial x} \implies \mathbf{X}_6 = \frac{\partial}{\partial x}. \text{ Hence, } e_6 = \mathbf{X}_6. \end{cases}$$

$$e_7 = \frac{\partial}{\partial y}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i, j, k = 1, 2, 4), \\ \eta^3 = R^3_1 q + R^3_2 x + R^3_3 y + R^3_4 z + S^3 + e^{wa^3_1}T^1 + e^{wa^3_2}T^2 + e^{wa^3_3}T^3 + e^{wa^3_4}T^4, \\ \eta^3 = 1 \implies S^3 = 1 \text{ and } R^3_1 = R^3_2 = R^3_3 = R^3_4 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_3 = \eta^3\frac{\partial}{\partial y} \implies \mathbf{X}_3 = \frac{\partial}{\partial y}. \text{ Hence, } e_7 = \mathbf{X}_7. \end{cases}$$

$$e_8 = q\frac{\partial}{\partial q}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k} T^k = 0, (i,j,k = 2,3,4), \\ \eta^1 = R^1_1 q + R^1_2 x + R^1_3 y + R^1_4 z + S^1 + e^{wa^1_1} T^1 + e^{wa^1_2} T^2 + e^{wa^1_3} T^3 + e^{wa^1_4} T^4, \\ \eta^1 = q \implies R^1_1 = 1 \text{ and } S^1 = R^1_2 = R^1_3 = R^1_4 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_8 = \eta^1 \frac{\partial}{\partial q} \implies \mathbf{X}_8 = q\frac{\partial}{\partial q}. \text{ Hence, } e_8 = \mathbf{X}_8. \end{cases}$$

$$e_9 = z\frac{\partial}{\partial z}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k} T^k = 0, (i,j,k = 1,2,3), \\ \eta^4 = R^4_1 q + R^4_2 x + R^4_3 y + R^4_4 z + S^4 + e^{wa^4_1} T^1 + e^{wa^4_2} T^2 + e^{wa^4_3} T^3 + e^{wa^4_4} T^4, \\ \eta^4 = z \implies R^4_4 = 1 \text{ and } S^4 = R^4_1 = R^4_2 = R^4_3 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_9 = \eta^4 \frac{\partial}{\partial z} \implies \mathbf{X}_9 = z\frac{\partial}{\partial z}. \text{ Hence, } e_9 = \mathbf{X}_9. \end{cases}$$

$$e_{10} = w\frac{\partial}{\partial t}\begin{cases} \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k} T^k = 0, (i,j,k = 1,..,4), \\ \xi = w \implies G = 1 \text{ and } K = L = 0, \\ \therefore \mathbf{X}_{10} = \xi \frac{\partial}{\partial t} \implies \mathbf{X}_{10} = w\frac{\partial}{\partial t}. \text{ Hence, } e_{10} = \mathbf{X}_{10}. \end{cases}$$

$$e_{11} = x\frac{\partial}{\partial x}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k} T^k = 0, (i,j,k = 1,3,4), \\ \eta^2 = R^2_1 q + R^2_2 x + R^2_3 y + R^2_4 z + S^2 + e^{wa^2_1} T^1 + e^{wa^2_2} T^2 + e^{wa^2_3} T^3 + e^{wa^2_4} T^4, \\ \eta^2 = x \implies R^2_2 = 1 \text{ and } S^2 = R^2_1 = R^2_3 = R^2_4 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_{11} = \eta^2 \frac{\partial}{\partial x} \implies \mathbf{X}_{11} = x\frac{\partial}{\partial x}. \text{ Hence, } e_{11} = \mathbf{X}_{11}. \end{cases}$$

$$e_{12} = y\frac{\partial}{\partial y}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k} T^k = 0, (i,j,k = 1,2,4), \\ \eta^3 = R^3_1 q + R^3_2 x + R^3_3 y + R^3_4 z + S^3 + e^{wa^3_1} T^1 + e^{wa^3_2} T^2 + e^{wa^3_3} T^3 + e^{wa^3_4} T^4, \\ \eta^3 = y \implies R^3_3 = 1 \text{ and } S^3 = R^3_1 = R^3_2 = R^3_4 = T^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_{12} = \eta^3 \frac{\partial}{\partial y} \implies \mathbf{X}_{12} = y\frac{\partial}{\partial y}. \text{ Hence, } e_{12} = \mathbf{X}_{12}. \end{cases}$$

$$e_{13} = e^{w}\frac{\partial}{\partial q}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i,j,k = 2,3,4), \\ \eta^1 = R^1_1 q + R^1_2 x + R^1_3 y + R^1_4 z + S^1 + e^{wa^1_1}T^1 + e^{wa^1_2}T^2 + e^{wa^1_3}T^3 + e^{wa^1_4}T^4, \\ \eta^1 = e^w \implies a^1_1 = T^1 = 1 \text{ and } R^1_1 = R^1_2 = R^1_3 = R^1_4 = S^1 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_{13} = \eta^1 \frac{\partial}{\partial q} \implies \mathbf{X}_{13} = e^w\frac{\partial}{\partial q}. \text{ Hence, } e_{13} = \mathbf{X}_{13}. \end{cases}$$

$$e_{14} = e^{cw}\frac{\partial}{\partial z}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i,j,k = 1,2,3), \\ \eta^4 = R^4_1 q + R^4_2 x + R^4_3 y + R^4_4 z + S^4 + e^{wa^4_1}T^1 + e^{wa^4_2}T^2 + e^{wa^4_3}T^3 + e^{wa^4_4}T^4, \\ \eta^4 = e^{cw} \implies a^4_4 = c,\ T^4 = 1 \text{ and } R^4_1 = R^4_2 = R^4_3 = R^4_4 = S^4 = T^2 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_{14} = \eta^4 \frac{\partial}{\partial z} \implies \mathbf{X}_{14} = e^{cw}\frac{\partial}{\partial z}. \text{ Hence, } e_{14} = \mathbf{X}_{14}. \end{cases}$$

$$e_{15} = e^{aw}\frac{\partial}{\partial x}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i,j,k = 1,3,4), \\ \eta^2 = R^2_1 q + R^2_2 x + R^2_3 y + R^2_4 z + S^2 + e^{wa^2_1}T^1 + e^{wa^2_2}T^2 + e^{wa^2_3}T^3 + e^{wa^2_4}T^4, \\ \eta^2 = e^{aw} \implies a^2_2 = a,\ T^2 = 1, \text{ and } R^2_1 = R^2_2 = R^2_3 = R^2_4 = S^2 = T^1 = T^3 = T^4 = 0, \\ \therefore \mathbf{X}_{14} = \eta^2 \frac{\partial}{\partial x} \implies \mathbf{X}_{15} = \frac{\partial}{\partial x}. \text{ Hence, } e_{15} = \mathbf{X}_{15}. \end{cases}$$

$$e_{16} = e^{bw}\frac{\partial}{\partial y}\begin{cases} \xi = 0, \eta = 0, \eta^i = R^i_j x^j + S^i + e^{wa^i_k}T^k = 0, (i,j,k = 1,2,4), \\ \eta^3 = R^3_1 q + R^3_2 x + R^3_3 y + R^3_4 z + S^3 + e^{wa^3_1}T^1 + e^{wa^3_2}T^2 + e^{wa^3_3}T^3 + e^{wa^3_4}T^4, \\ \eta^3 = e^{bw} \implies a^3_3 = b,\ T^3 = 1, \text{ and } R^3_1 = R^3_2 = R^3_3 = R^3_4 = S^3 = T^1 = T^2 = T^4 = 0, \\ \therefore \mathbf{X}_{16} = \eta^3 \frac{\partial}{\partial y} \implies \mathbf{X}_{16} = e^{bw}\frac{\partial}{\partial y}. \text{ Hence, } e_{16} = \mathbf{X}_{16}. \end{cases}$$

We summarize the above results in the following Proposition.

Proposition 5.1. *The symmetry vector fields associated with the geodesic equations of $A_{5,7}$ (4.2) are obtained from the Lie symmetry vector field $\mathbf{X} = \xi(t, x^i, w)\frac{\partial}{\partial t} + \eta^i(t, x^i, w)\frac{\partial}{\partial x^i} + \eta(t, x^i, w)\frac{\partial}{\partial w}$ of (5.3), that is, $\ddot{x}^i = a^i_j \dot{x}^j \dot{w},\ \ddot{w} = 0$.*

Appendix A

NOTATION

DEs	Differential Equations
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
	n-dimensional vector space
G	Group
I	Identity element
$\mathfrak{g}$	Lie Algebra
$\mathfrak{sl}(n, \mathbb{R})$	Special linear Lie algebra of order n
$[\,,\,]$	Lie bracket
$\oplus$	Direct sum of vector spaces
$\rtimes$	Semi-direct product of vector spaces
char $\mathbb{F}$	Characteristic of $\mathbb{F}$
δ_i^j	Kronecker delta with $\delta_i^j = 1$ for $i = j$ and $\delta_i^j = 0$ for $i \neq j$
det	Determinant of a squar matrix
ϵ	Real parameter
$A_{m,n}$	m denotes the dimension of the algebra and n the nth one in the list
D	Total derivative with respect to x

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